A solid understanding of lines and their equations is fundamental to a study of differential calculus. The Connections to Calculus feature for Chapter 1 reviews these essential concepts and skills, and provides an opportunity for practice in the context of a future calculus course.
In everyday life, we encounter a large variety of relationships. For instance, the time it takes us to get to work is related to our average speed; the monthly cost of heating a home is related to the average outdoor temperature; and in many cases, the amount of our charitable giving is related to changes in the cost of living. In each case we say that a relation exists between the two quantities.

A. Relations, Mapping Notation, and Ordered Pairs

In the most general sense, a relation is simply a correspondence between two sets. Relations can be represented in many different ways and may even be very "unmathematical," like the one shown in Figure 1.1 between a set of people and the set of their corresponding birthdays. If $P$ represents the set of people and $B$ represents the set of birthdays, we say that elements of $P$ correspond to elements of $B$, or the birthday relation maps elements of $P$ to elements of $B$. Using what is called mapping notation, we might simply write $P \rightarrow B$. From a purely practical standpoint, we note that while it is possible for two different people to share the same birthday, it is quite impossible for the same person to have two different birthdays. Later, this observation will help us mark the difference between a relation and special kind of relation called a function.

The bar graph in Figure 1.2 is also an example of a relation. In the graph, each year is related to annual consumer spending per person on cable and satellite television. As an alternative to mapping or a bar graph, this relation could also be represented using ordered pairs. For example, the ordered pair $(3, 234)$ would indicate that in 2005, spending per person on cable and satellite TV in the United States averaged $234. When a relation is represented using ordered pairs, we say the relation is pointwise-defined.

Over a long period of time, we could collect many ordered pairs of the form $(t, s)$, where consumer spending $s$ depends on the time $t$. For this reason we often call the second coordinate of an ordered pair (in this case $s$) the dependent variable, with the first coordinate designated as the independent variable. The set of all first coordinates is called the domain of the relation. The set of all second coordinates is called the range.
### Example 1: Expressing a Relation as a Mapping and as a Pointwise-Defined Relation

Represent the relation from Figure 1.2 in mapping notation and as a pointwise-defined relation, then state its domain and range.

Let $t$ represent the year and $s$ represent consumer spending. The mapping $t \rightarrow s$ gives the diagram shown. As a pointwise-defined relation we have $(3, 192), (5, 234), (7, 281), (9, 375),$ and $(11, 411)$. The domain is the set $\{3, 5, 7, 9, 11\}$; the range is $\{192, 234, 281, 375, 411\}$.

For more on this relation, see Exercise 93.

### B. The Graph of a Relation

Relations can also be stated in equation form. The equation $y = x - 1$ expresses a relation where each $y$-value is one less than the corresponding $x$-value (see Table 1.1). The equation $x = |y|$ expresses a relation where each $x$-value corresponds to the absolute value of $y$ (see Table 1.2). In each case, the relation is the set of all ordered pairs $(x, y)$ that create a true statement when substituted, and a few ordered pair solutions are shown in the tables for each equation.

Relations can be expressed graphically using a rectangular coordinate system. It consists of a horizontal number line (the $x$-axis) and a vertical number line (the $y$-axis) intersecting at their zero marks. The point of intersection is called the origin. The $x$- and $y$-axes create a flat, two-dimensional surface called the $xy$-plane and divide the plane into four regions called quadrants. These are labeled using a capital "Q" (for quadrant) and the Roman numerals I through IV, beginning in the upper right and moving counterclockwise (Figure 1.3). The grid lines shown denote the integer values on each axis and further divide the plane into a coordinate grid, where every point in the plane corresponds to an ordered pair. Since a point at the origin has not moved along either axis, it has coordinates $(0, 0).$ To plot a point $(x, y)$ means we place a dot at its location in the $xy$-plane. A few of the ordered pairs from $y = x - 1$ are plotted in Figure 1.4, where a noticeable pattern emerges—-the points seem to lie along a straight line.

If a relation is pointwise-defined, the graph of the relation is simply the plotted points. The graph of a relation in equation form, such as $y = x - 1$, is the set of all ordered pairs $(x, y)$ that are solutions (make the equation true).

### Solutions to an Equation in Two Variables

1. If substituting $x = a$ and $y = b$ results in a true equation, the ordered pair $(a, b)$ is a solution and on the graph of the relation.

2. If the ordered pair $(a, b)$ is on the graph of a relation, it is a solution (substituting $x = a$ and $y = b$ will result in a true equation).
We generally use only a few select points to determine the shape of a graph, then draw a straight line or smooth curve through these points, as indicated by any patterns formed.

**EXAMPLE 2**  
**Graphing Relations**  
Graph the relations \( y = x - 1 \) and \( x = |y| \) using the ordered pairs given in Tables 1.1 and 1.2.

**Solution**  
For \( y = x - 1 \), we plot the points then connect them with a straight line (Figure 1.5). For \( x = |y| \), the plotted points form a V-shaped graph made up of two half lines (Figure 1.6).

![Graphs](image)

**WORTHY OF NOTE**  
As the graphs in Example 2 indicate, arrowheads are used where appropriate to indicate the infinite extension of a graph.

While we used only a few points to graph the relations in Example 2, they are actually made up of an infinite number of ordered pairs that satisfy each equation, including those that might be rational or irrational. This understanding is an important part of reading and interpreting graphs, and is illustrated for you in Figures 1.7 through 1.10.

![Graphs](image)
Since there are an infinite number of ordered pairs forming the graph of \( y = x - 1 \), the domain cannot be given in list form. Here we note \( x \) can be any real number and write \( D: x \in \mathbb{R} \). Likewise, \( y \) can be any real number and for the range we have \( R: y \in \mathbb{R} \). All of these points together make these graphs continuous, which for our purposes means you can draw the entire graph without lifting your pencil from the paper.

Actually, a majority of graphs cannot be drawn using only a straight line or directed line segments. In these cases, we rely on a “sufficient number” of points to outline the basic shape of the graph, then connect the points with a smooth curve. As your experience with graphing increases, this “sufficient number of points” tends to get smaller as you learn to anticipate what the graph of a given relation should look like. In particular, for the linear graph in Figure 1.5 we notice that both the \( x \)- and \( y \)-variables have an implied exponent of 1. This is in fact a characteristic of linear equations and graphs. In Example 3 we’ll notice that if the exponent on one of the variables is 2 (either \( x \) or \( y \) is squared) while the other exponent is 1, the result is a graph called a parabola. If the \( x \)-term is squared (Example 3a) the parabola is oriented vertically, as in Figure 1.11, and its highest or lowest point is called the vertex. If the \( y \)-term is squared (Example 3c), the parabola is oriented horizontally, as in Figure 1.13, and the leftmost or rightmost point is the vertex. The graphs and equations of other relations likewise have certain identifying characteristics. See Exercises 85 through 92.

**EXAMPLE 3**

Graphing Relations

Graph the following relations by completing the tables given. Then use the graph to state the domain and range of the relation.

\( a. \quad y = x^2 - 2x \quad b. \quad y = \sqrt{9 - x^2} \quad c. \quad x = y^2 \)

**Solution**

For each relation, we use each \( x \)-input in turn to determine the related \( y \)-output(s), if they exist. Results can be entered in a table and the ordered pairs used to assist in drawing a complete graph.

\( a. \quad y = x^2 - 2x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>((x, y)) Ordered Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>24</td>
<td>(-4, 24)</td>
</tr>
<tr>
<td>-3</td>
<td>15</td>
<td>(-3, 15)</td>
</tr>
<tr>
<td>-2</td>
<td>8</td>
<td>(-2, 8)</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
<td>(-1, 3)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>(1, -1)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>(2, 0)</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>(3, 3)</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>(4, 8)</td>
</tr>
</tbody>
</table>

The resulting vertical parabola is shown in Figure 1.11. Although \((-4, 24)\) and \((-3, 15)\) cannot be plotted here, the arrowheads indicate an infinite extension of the graph, which will include these points. This “infinite extension” in the upward direction shows there is no largest \( y \)-value (the graph becomes infinitely “tall”). Since the smallest possible \( y \)-value is \(-1\) [from the vertex \((1, -1)\)], the range is \( y \geq -1 \). However, this extension also continues forever in the outward direction as well (the graph gets wider and wider). This means the \( x \)-value of all possible ordered pairs could vary from negative to positive infinity, and the domain is all real numbers. We then have \( D: x \in \mathbb{R} \) and \( R: y \geq -1 \).
b.  \[ y = \sqrt{9 - x^2} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>(( x, y ))</th>
<th>Ordered Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>not real</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td>0</td>
<td>(-3, 0)</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td>( \sqrt{5} )</td>
<td>(-2, ( \sqrt{5} ))</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>( 2\sqrt{2} )</td>
<td>(-1, ( 2\sqrt{2} ))</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td>(0, 3)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>( 2\sqrt{2} )</td>
<td>(1, ( 2\sqrt{2} ))</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \sqrt{5} )</td>
<td>(2, ( \sqrt{5} ))</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>(3, 0)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>not real</td>
<td>—</td>
<td></td>
</tr>
</tbody>
</table>

The result is the graph of a **semicircle** (Figure 1.12). The points with irrational coordinates were graphed by estimating their location. Note that when \( x < -3 \) or \( x > 3 \), the relation \( y = \sqrt{9 - x^2} \) does not represent a real number and no points can be graphed. Also note that no arrowheads are used since the graph terminates at \((-3, 0)\) and \((3, 0)\). These observations and the graph itself show that for this relation, \( D: -3 \leq x \leq 3 \), and \( R: 0 \leq y \leq 3 \).

c. Similar to \( x = |y| \), the relation \( x = y^2 \) is defined only for \( x \geq 0 \) since \( y^2 \) is always nonnegative \((-1 = y^2 \) has no real solutions). In addition, we reason that each positive \( x \)-value will correspond to two \( y \)-values. For example, given \( x = 4 \), \((4, -2)\) and \((4, 2)\) are both solutions to \( x = y^2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>(( x, y ))</th>
<th>Ordered Pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>not real</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>not real</td>
<td>—</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>(0, 0)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1, 1</td>
<td>(1, -1) and (1, 1)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( -\sqrt{2}, \sqrt{2} )</td>
<td>(2, ( -\sqrt{2} )) and (2, ( \sqrt{2} ))</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( -\sqrt{3}, \sqrt{3} )</td>
<td>(3, ( -\sqrt{3} )) and (3, ( \sqrt{3} ))</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>-2, 2</td>
<td>(-2, 2) and (4, 2)</td>
<td></td>
</tr>
</tbody>
</table>

This relation is a horizontal parabola, with a vertex at \((0, 0)\) (Figure 1.13). The graph begins at \( x = 0 \) and extends infinitely to the right, showing the domain is \( x \geq 0 \). Similar to Example 3a, this “infinite extension” also extends in both the upward and downward directions and the \( y \)-value of all possible ordered pairs could vary from negative to positive infinity. We then have \( D: x \geq 0 \) and \( R: y \in \mathbb{R} \).

\[ \checkmark \] B. You've just seen how we can graph relations

Now try Exercises 17 through 24

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C. **Graphing Relations on a Calculator**

For relations given in equation form, the **TABLE** feature of a graphing calculator can be used to compute ordered pairs, and the **GRAPH** feature to draw the related graph. To use these features, we first solve the equation for the variable \( y \) (write \( y \) in terms of \( x \)), then enter the right-hand expression on the calculator's **GRAPH** (equation editor) screen.
We can then select either the \emph{Graph} feature, or set-up, create, and use the \emph{Table} feature. We’ll illustrate here using the relation $-2x + y = 3$.

1. Solve for $y$ in terms of $x$.
   \[
   -2x + y = 3 \quad \text{given equation}
   \]
   \[
   y = 2x + 3 \quad \text{add } 2x \text{ to each side}
   \]

2. Enter the equation.
   Press the \text{Y=} key to access the equation editor, then enter $2x + 3$ as $Y_1$ (see Figure 1.14). The calculator automatically highlights the equal sign, showing that equation $Y_1$ is now active. If there are other equations on the screen, you can either \text{clear} them or deactivate them by moving the cursor to overlay the equal sign and pressing \text{Clear}.

3. Use the \emph{Table} or \text{2nd} \emph{(Graph)} (TBLSET).
   To set up the table, we use the keystrokes \text{2nd} \emph{(Graph)} (TBLSET). For this exercise, we’ll put the table in the “\emph{Indpnt: Auto Ask}” mode, which will have the calculator automatically generate the input and output values. In this mode, we can tell the calculator where to start the inputs (we chose \text{TblStart = -3}), and have the calculator produce the input values using any increment desired (we choose \text{ΔTbl = 1}). See Figure 1.15A. Access the table using \text{2nd} \emph{(Graph)} (TABLE), and the table resulting from this setup is shown in Figure 1.15B. Notice that all ordered pairs satisfy the equation $y = 2x + 3$, or “$y$ is twice $x$ increased by 3.”

Since much of our graphical work is centered at $(0, 0)$ on the coordinate grid, the calculator’s default settings for the standard viewing \text{Window} are $[-10, 10]$ for both $x$ and $y$ (Figure 1.16). The \text{Xscl} and \text{Yscl} values give the scale used on each axis, and indicate here that each “tick mark” will be 1 unit apart. To graph the line in this window, we can use the \text{Zoom} key and select \text{6:ZStandard} (Figure 1.17), which resets the window to these default settings and automatically graphs the line (Figure 1.18).

In addition to using the calculator’s \emph{Table} feature to find ordered pairs for a given graph, we can also use the calculator’s \text{Trace} feature. As the name implies, this feature allows us to “trace” along the graph by moving a cursor to the left and right \text{Trace} using the arrow keys. The calculator displays the coordinates of the cursor’s location each time it moves. After pressing the \text{Trace} key, the marker appears automatically and as you move it to the left or right, the current coordinates are shown at the bottom.
of the screen (Figure 1.19). While not very “pretty,” \((-0.8510638, 1.2978723)\) is a point on this line (rounded to seven decimal places) and satisfies its equation. The calculator is displaying these decimal values because the viewing screen is exactly 95 pixels wide, 47 pixels to the left of the y-axis, and 47 pixels to the right. This means that each time you press the left or right arrow, the x-value changes by 1/47th—which is not a nice round number. To have the calculator through “friendlier” values, we can use the \(4:Z\text{Decimal}\) feature, which sets \(X\text{min} = -4.7\) and \(X\text{max} = 4.7\), or \(8:Z\text{Integer}\), which sets \(X\text{min} = -47\) and \(X\text{max} = 47\). Let’s use the \(4:Z\text{Decimal}\) option here, noting the calculator automatically regraphs the line. Pressing the \(y=\) key once again and moving the marker shows that more “friendly” ordered pair solutions are displayed (Figure 1.20). Other methods for finding a friendly window are discussed later in this section.

**EXAMPLE 4**

Graphing a Relation Using Technology

Use a calculator to graph \(2x + 3y = -6\). Then use the **TABLE** feature to determine the value of \(y\) when \(x = 0\), and the value of \(x\) when \(y = 0\). Write each result in ordered pair form.

**Solution**

We begin by solving the equation for \(y\), so we can enter it on the \(\text{graph}\) screen.

\[
2x + 3y = -6 \quad \text{given equation}
\]

\[
3y = -2x - 6 \quad \text{subtract } 2x \text{ (isolate the } y\text{-term)}
\]

\[
y = -\frac{2}{3}x - 2 \quad \text{divide by 3}
\]

Entering \(y = -\frac{2}{3}x - 2\) on the \(\text{graph}\) screen and using \(6:Z\text{Standard}\) produces the graph shown. Using the **TABLE** and scrolling as needed, shows that when \(x = 0\), \(y = -2\), and when \(y = 0\), \(x = -3\). As ordered pairs we have \((0, -2)\) and \((-3, 0)\).

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**C.** You’ve just seen how we can graph a relation using a calculator

**D. The Equation and Graph of a Circle**

Using the midpoint and distance formulas, we can develop the equation of another important relation, that of a circle. As the name suggests, the **midpoint of a line segment** is located halfway between the endpoints. On a standard number line, the midpoint of the line segment with endpoints 1 and 5 is 3, but more important, note that 3 is the
average distance (from zero) of 1 unit and 5 units: \( \frac{1 + 5}{2} = \frac{6}{2} = 3 \). This observation can be extended to find the midpoint between any two points \((x_1, y_1)\) and \((x_2, y_2)\) in the xy-plane. We simply find the average distance between the x-coordinates and the average distance between the y-coordinates.

**The Midpoint Formula**

Given any line segment with endpoints \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\), the midpoint \(M\) is given by

\[
M: \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)
\]

The midpoint formula can be used in many different ways. Here we’ll use it to find the coordinates of the center of a circle.

**Example 5**

**Using the Midpoint Formula**

The diameter of a circle has endpoints at \(P_1 = (-3, -2)\) and \(P_2 = (5, 4)\). Use the midpoint formula to find the coordinates of the center, then plot this point.

**Solution**

Midpoint: \(M: \left( \frac{-3 + 5}{2}, \frac{-2 + 4}{2} \right)\)

\[
M: \left( \frac{2}{2}, \frac{2}{2} \right) = (1, 1)
\]

The center is at \((1, 1)\), which we graph directly on the diameter as shown.

**Now try Exercises 29 through 38**

**The Distance Formula**

In addition to a line segment’s midpoint, we are often interested in the length of the segment. For any two points \((x_1, y_1)\) and \((x_2, y_2)\) not lying on a horizontal or vertical line, a right triangle can be formed as in Figure 1.21. Regardless of the triangle’s orientation, the length of side \(a\) (the horizontal segment or base of the triangle) will have length \(|x_2 - x_1|\) units, with side \(b\) (the vertical segment or height) having length \(|y_2 - y_1|\) units. From the Pythagorean theorem (Appendix A.6), we see that \(c^2 = a^2 + b^2\) corresponds to \(c^2 = (|x_2 - x_1|)^2 + (|y_2 - y_1|)^2\). By taking the square root of both sides we obtain the length of the hypotenuse, which is identical to the distance between these two points: \(c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\). The result is called the distance formula, although it’s most often written using \(d\) for distance, rather than \(c\). Note the absolute value bars are dropped from the formula, since the square of any quantity is always nonnegative. This also means that *either* point can be used as the initial point in the computation.

**The Distance Formula**

Given any two points \(P_1 = (x_1, y_1)\) and \(P_2 = (x_2, y_2)\), the straight line distance \(d\) between them is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]
EXAMPLE 6 Using the Distance Formula

Use the distance formula to find the diameter of the circle from Example 5.

Solution For \((x_1, y_1) = (-3, -2)\) and \((x_2, y_2) = (5, 4)\), the distance formula gives

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
= \sqrt{[5 - (-3)]^2 + [4 - (-2)]^2} \\
= \sqrt{8^2 + 6^2} \\
= \sqrt{100} = 10
\]

The diameter of the circle is 10 units long.

Now try Exercises 39 through 48

A circle can be defined as the set of all points in a plane that are a fixed distance called the radius, from a fixed point called the center. Since the definition involves distance, we can construct the general equation of a circle using the distance formula. Assume the center has coordinates \((h, k)\), and let \((x, y)\) represent any point on the graph. The distance between these points is equal to the radius \(r\), and the distance formula yields:

\[
\sqrt{(x - h)^2 + (y - k)^2} = r
\]

Squaring both sides gives the equation of a circle in standard form:

\[
(x - h)^2 + (y - k)^2 = r^2
\]

The Equation of a Circle

A circle of radius \(r\) with center at \((h, k)\) has the equation

\[
(x - h)^2 + (y - k)^2 = r^2
\]

If \(h = 0\) and \(k = 0\), the circle is centered at \((0, 0)\) and the graph is a central circle with equation \(x^2 + y^2 = r^2\). At other values for \(h\) or \(k\), the center is at \((h, k)\) with no change in the radius. Note that an open dot is used for the center, as it’s actually a point of reference and not a part of the graph.

EXAMPLE 7 Finding the Equation of a Circle in Standard Form

Find the equation of a circle with center \((0, -1)\) and radius 4.

Solution Since the center is at \((0, -1)\) we have \(h = 0, k = -1,\) and \(r = 4\). Using the standard form \((x - h)^2 + (y - k)^2 = r^2\) we obtain

\[
(x - 0)^2 + [y - (-1)]^2 = 4^2 \\
x^2 + (y + 1)^2 = 16
\]

Substitute 0 for \(h\), -1 for \(k\), and 4 for \(r\) simplify
The graph of \( x^2 + (y + 1)^2 = 16 \) is shown in the figure.

Now try Exercises 49 through 66

The graph of a circle can be obtained by first identifying the coordinates of the center and the length of the radius from the equation in standard form. After plotting the center point, we count a distance of \( r \) units left and right of center in the horizontal direction, and up and down from center in the vertical direction, obtaining four points on the circle. Neatly graph a circle containing these four points.

**EXAMPLE 8**

**Graphing a Circle**

Graph the circle represented by \((x - 2)^2 + (y + 3)^2 = 12\). Clearly label the center and radius.

**Solution**

Comparing the given equation with the standard form, we find the center is at \((2, -3)\) and the radius is \(r = 2\sqrt{3} \approx 3.5\).

\[
(x - h)^2 + (y - k)^2 = r^2 \quad \text{standard form}
\]

\[
(x - 2)^2 + (y + 3)^2 = 12 \quad \text{given equation}
\]

\[
-h = -2 \quad -k = 3 \quad r^2 = 12
\]

\[
h = 2 \quad k = -3 \quad r = \sqrt{12} = 2\sqrt{3} = 3.5 \quad \text{radius must be positive}
\]

Plot the center \((2, -3)\) and count approximately 3.5 units in the horizontal and vertical directions. Complete the circle by freehand drawing or using a compass. The graph shown is obtained.

Circle

- Center: \((2, -3)\)
- Radius: \(r = 2\sqrt{3}\)

Endpoints of horizontal diameter

- \((2 - 2\sqrt{3}, -3)\) and \((2 + 2\sqrt{3}, -3)\)

Endpoints of vertical diameter

- \((2, -3 + 2\sqrt{3})\) and \((2, -3 - 2\sqrt{3})\)

Now try Exercises 67 through 72
In Example 8, note the equation is composed of binomial squares in both $x$ and $y$. By expanding the binomials and collecting like terms, we can write the equation of the circle in **general form**:

\[
(x - 2)^2 + (y + 3)^2 = 12 \quad \text{standard form}
\]
\[
x^2 - 4x + 4 + y^2 + 6y + 9 = 12 \quad \text{expand binomials}
\]
\[
x^2 + y^2 - 4x + 6y + 1 = 0 \quad \text{combine like terms—general form}
\]

For future reference, observe the general form contains a sum of second-degree terms in $x$ and $y$, and that both terms have the same coefficient (in this case, "1").

Since this form of the equation was derived by squaring binomials, it seems reasonable to assume we can go back to the standard form by creating binomial squares in $x$ and $y$. This is accomplished by **completing the square**.

### Example 9

**Finding the Center and Radius of a Circle**

Find the center and radius of the circle with equation $x^2 + y^2 + 2x - 4y - 4 = 0$. Then sketch its graph and label the center and radius.

**Solution**

To find the center and radius, we complete the square in both $x$ and $y$.

\[
x^2 + y^2 + 2x - 4y - 4 = 0 \quad \text{given equation}
\]
\[
(x^2 + 2x + \_\_) + (y^2 - 4y + \_) = 4 \quad \text{group $x$-terms and $y$-terms; add 4}
\]
\[
(x^2 + 2x + 1) + (y^2 - 4y + 4) = 4 + 1 + 4 \quad \text{complete each binomial square}
\]
\[
\text{adds 1 to left side adds 4 to left side add 1 + 4 to right side}
\]
\[
(x + 1)^2 + (y - 2)^2 = 9 \quad \text{factor and simplify}
\]

The center is at $(-1, 2)$ and the radius is $r = \sqrt{9} = 3$.

Circle
Center: $(-1, 2)$
Radius: $r = 3$

Now try Exercises 73 through 84

### Example 10

**Applying the Equation of a Circle**

To aid in a study of nocturnal animals, some naturalists install a motion detector near a popular watering hole. The device has a range of 10 m in any direction. Assume the water hole has coordinates $(0, 0)$ and the device is placed at $(2, -1)$.

**a.** Write the equation of the circle that models the maximum effective range of the device.

**b.** Use the distance formula to determine if the device will detect a badger that is approaching the water and is now at coordinates $(11, -5)$. 
Solution

a. Since the device is at (2, -1) and the radius (or reach) of detection is 10 m, any movement in the interior of the circle defined by \((x - 2)^2 + (y + 1)^2 = 10^2\) will be detected.

b. Using the points (2, -1) and (11, -5) in the distance formula yields:

\[
\begin{align*}
  d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
  &= \sqrt{(11 - 2)^2 + (-5 - (-1))^2} \\
  &= \sqrt{9^2 + (-4)^2} \\
  &= \sqrt{81 + 16} \\
  &= \sqrt{97} \approx 9.85
\end{align*}
\]

Since 9.85 < 10, the badger is within range of the device and will be detected.

Now try Exercises 95 through 100

When using a graphing calculator to study circles, it’s important to note that most standard viewing windows have the x- and y-values preset at \([-10, 10]\) even though the calculator screen is not square. This tends to compress the y-values and give a skewed image of the graph. If the circle appears oval in shape, use the arrow command to obtain the correct perspective. Graphing calculators can produce the graph of a circle in various ways, and the choice of method simply depends on what you’d like to accomplish. To simply view the graph or compare two circular graphs, the DRAW command is used. From the home screen press: DRAW (DRAW 9:Circle). This generates the “Circle(“ command, with the left parentheses indicating we need to supply three inputs, separated by commas. These inputs are the x-coordinate of the center, the y-coordinate of the center, and the radius of the circle. For the circle defined by the equation \((x - 3)^2 + (y + 2)^2 = 49\), we know the center is at (3, -2) and the radius is 7 units. The resulting command and graph are shown in Figures 1.22 and 1.23.

While the DRAW command will graph any circle, we are unable to use the TRACE or CALC commands to interact with the graph. To make these features available, we must first solve for x in terms of y, as we did previously (the 1:ClrDraw command is used to clear the graph). Consider the relation \(x^2 + y^2 = 25\), which we know is the equation of a circle centered at (0, 0) with radius \(r = 5\).

\[
\begin{align*}
  x^2 + y^2 &= 25 & \text{original equation} \\
  y^2 &= 25 - x^2 & \text{isolate } y^2 \\
  y &= \pm \sqrt{25 - x^2} & \text{solve for } y
\end{align*}
\]

Note that we can separate this result into two parts, enabling the calculator to graph \(Y_1 = \sqrt{25 - x^2}\) (giving the “upper half” of the circle), and \(Y_2 = -\sqrt{25 - x^2}\) (giving the “lower half”). Enter these on the \(Y=\) screen (note that \(Y_2 = -Y_1\) can be used instead of reentering the entire expression: \(Y_2 = -Y_1\)). If we graph \(Y_1\) and \(Y_2\) on the standard screen, the result appears more oval than circular (Figure 1.24). Using the arrow option, the tick marks become equally spaced on both axes (Figure 1.25).
D. You've just seen how we can develop the equation and graph of a circle using the distance and midpoint formulas.

Although it is a much improved graph, the circle does not appear "closed" as the calculator lacks sufficient pixels to show the proper curvature. A second alternative is to manually set a "friendly" window. Using \( \text{Xmin} = -9.4 \), \( \text{Xmax} = 9.4 \), \( \text{Ymin} = -6.2 \), and \( \text{Ymax} = 6.2 \) will generate a better graph due to the number of pixels available. Note that we can jump between the upper and lower halves of the circle using the up or down arrows. See Exercises 101 and 102.

### 1.1 EXERCISES

#### CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. If a relation is defined by a set of ordered pairs, the domain is the set of all _______ components, the range is the set of all _______ components.

2. For the equation \( y = x + 5 \) and the ordered pair \((x, y)\), \(x\) is referred to as the input or _______ variable, while \(y\) is called the _______ or dependent variable.

3. A circle is defined as the set of all points that are an equal distance, called the _______, from a given point, called the _______.

4. For \( x^2 + y^2 = 25 \), the center of the circle is at _______ and the length of the radius is _______ units. The graph is called a _______ circle.

5. Discuss/Explain how to find the center and radius of the circle defined by the equation \( x^2 + y^2 - 6x = 7 \). How would this circle differ from the one defined by \( x^2 + y^2 - 6y = 7 \)?

6. In Example 3(b) we graphed the semicircle defined by \( y = \sqrt{9 - x^2} \). Discuss how you would obtain the equation of the full circle from this equation, and how the two equations are related.

#### DEVELOPING YOUR SKILLS

Represent each relation in mapping notation, then state the domain and range.

7. [Graph of GPA vs. Year in college]

8. [Graph of Efficiency rating vs. Month]

State the domain and range of each pointwise-defined relation.

9. \{\((1, 2), (3, 4), (5, 6), (7, 8), (9, 10)\)\}

10. \{\((-2, 4), (-3, -5), (-1, 3), (4, -5), (2, -3)\)\}

11. \{\((4, 0), (-1, 5), (2, 4), (4, 2), (-3, 3)\)\}

12. \{\((-1, 1), (0, 4), (2, -5), (-3, 4), (2, 3)\)\}

Complete each table using the given equation. For Exercises 15, 16, 21, and 22, each input may correspond to two outputs (be sure to find both if they exist). Use these points to graph the relation. For Exercises 17 through 24, also state the domain and range.

13. \( y = \frac{2}{3}x + 1 \)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-6</td>
<td></td>
</tr>
<tr>
<td>-3</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

14. \( y = \frac{5}{4}x + 3 \)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
15. \( x + 2 = |y| \)  
\[
\begin{array}{c|c|c|c|c|c|c|c}
| x | & -2 & 0 & 1 & 3 & 5 & 6 & 7 \\
| y | &    &    &    &    &    &    &    \\
\end{array}
\]

16. \( |y + 1| = x \)  
\[
\begin{array}{c|c|c|c|c|c|c|c}
| x | & 0 & 1 & 3 & 5 & 6 & 7 \\
| y | &    &    &    &    &    &    \\
\end{array}
\]

17. \( y = x^2 - 1 \)  
\[
\begin{array}{c|c|c|c|c|c}
| x | & -3 & 2 & 4 & 12 & 3 \\
| y | &    &    &    &    &    \\
\end{array}
\]

18. \( y = -x^2 + 3 \)  
\[
\begin{array}{c|c|c|c|c|c|c}
| x | & -2 & 1 & 2 & 3 & 4 \\
| y | &    &    &    &    &    \\
\end{array}
\]

19. \( y = \sqrt{25 - x^2} \)  
\[
\begin{array}{c|c|c|c|c|c|c}
| x | & -4 & -3 & 0 & 2 & 3 & 4 \\
| y | &    &    &    &    &    &    \\
\end{array}
\]

20. \( y = \sqrt{169 - x^2} \)  
\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
| x | & -12 & -5 & 0 & 3 & 5 & 12 \\
| y | &    &    &    &    &    &    \\
\end{array}
\]

21. \( x - 1 = y^2 \)  
\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
| x | & 10 & 5 & 4 & 2 & 1.25 & 1 \\
| y | &    &    &    &    &    &    \\
\end{array}
\]

22. \( y^2 - 2 = x \)  
\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
| x | & -2 & -1 & 0 & 1 & 2 & 3 \\
| y | &    &    &    &    &    &    \\
\end{array}
\]

23. \( y = \sqrt{x + 1} \)  
\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
| x | & -9 & -2 & -1 & 0 & 4 & 7 \\
| y | &    &    &    &    &    &    \\
\end{array}
\]

24. \( y = (x - 1)^3 \)  
\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
| x | & -2 & -1 & 0 & 1 & 2 & 3 \\
| y | &    &    &    &    &    &    \\
\end{array}
\]

Use a graphing calculator to graph the following relations. Then use the TABLE feature to determine the value of \( y \) when \( x = 0 \), and the value(s) of \( x \) when \( y = 0 \), and write the results in ordered pair form.

25. \(-2x + 5y = 10\)
26. \(x + 2y = 6\)
27. \(y = x^2 - 4x\)
28. \(y + x^2 = 2x + 3\)

Find the midpoint of each segment with the given endpoints.

29. \((1, 8), (5, -6)\)
30. \((5, 6), (6, -8)\)
31. \((-4.5, 9.2), (3.1, -9.8)\)
32. \((5.2, 7.1), (6.3, -7.1)\)

33. \((\frac{1}{5}, \frac{2}{3}); (\frac{-1}{10}, \frac{3}{4})\)
34. \((\frac{3}{4}, \frac{1}{3}); (\frac{3}{5}, \frac{1}{6})\)

Find the midpoint of each segment.

35. \[\text{Diagram of two lines meeting at a point.}\]
36. \[\text{Diagram of two lines meeting at a point.}\]

Find the center of each circle with the diameter shown.

37. \[\text{Diagram of a circle with center and diameter marked.}\]
38. \[\text{Diagram of a circle with center and diameter marked.}\]

39. Use the distance formula to find the length of the line segment in Exercise 35.
40. Use the distance formula to find the length of the line segment in Exercise 36.
41. Use the distance formula to find the length of the diameter of the circle in Exercise 37.
42. Use the distance formula to find the length of the diameter of the circle in Exercise 38.
In Exercises 43 to 48, three points that form the vertices of a triangle are given. Use the distance formula to determine if any of the triangles are right triangles (the three sides satisfy the Pythagorean Theorem $a^2 + b^2 = c^2$).

43. $(−3, 7), (2, 2), (5, 5)$
44. $(7, 0), (−1, 0), (7, 4)$
45. $(−4, 3), (−7, −1), (3, −2)$
46. $(5, 2), (0, −3), (4, −4)$
47. $(−3, 2), (−1, 5), (−6, 4)$
48. $(0, 0), (−5, 2), (2, −5)$

Find the equation of a circle satisfying the conditions given, then sketch its graph.

49. center $(0, 0)$, radius 3
50. center $(0, 0)$, radius 6
51. center $(5, 0)$, radius $\sqrt{3}$
52. center $(0, 4)$, radius $\sqrt{5}$
53. center $(4, −3)$, radius 2
54. center $(3, −8)$, radius 9
55. center $(−7, −4)$, radius $\sqrt{7}$
56. center $(−2, −5)$, radius $\sqrt{6}$
57. center $(1, −2)$, diameter 6
58. center $(−2, 3)$, diameter 10
59. center $(4, 5)$, diameter $4\sqrt{3}$
60. center $(5, 1)$, diameter $4\sqrt{5}$
61. center at $(7, 1)$, graph contains the point $(1, −7)$
62. center at $(−8, 3)$, graph contains the point $(−3, 15)$
63. center at $(3, 4)$, graph contains the point $(7, 9)$
64. center at $(−5, 2)$, graph contains the point $(−1, 3)$
65. diameter has endpoints $(5, 1)$ and $(5, 7)$
66. diameter has endpoints $(2, 3)$ and $(8, 3)$

Identify the center and radius of each circle, then graph. Also state the domain and range of the relation.

67. $(x − 2)^2 + (y − 3)^2 = 4$
68. $(x − 5)^2 + (y − 1)^2 = 9$
69. $(x + 1)^2 + (y − 2)^2 = 12$
70. $(x − 7)^2 + (y + 4)^2 = 20$
71. $(x + 4)^2 + y^2 = 81$
72. $x^2 + (y − 3)^2 = 49$

Write each equation in standard form to find the center and radius of the circle. Then sketch the graph.

73. $x^2 + y^2 − 10x − 12y + 4 = 0$
74. $x^2 + y^2 + 6x − 8y − 6 = 0$
75. $x^2 + y^2 − 10x + 4y + 4 = 0$
76. $x^2 + y^2 + 6x + 4y + 12 = 0$
77. $x^2 + y^2 + 6y − 5 = 0$
78. $x^2 + y^2 − 8x + 12 = 0$
79. $x^2 + y^2 + 4x + 10y + 18 = 0$
80. $x^2 + y^2 − 8x − 14y − 47 = 0$
81. $x^2 + y^2 + 14x + 12 = 0$
82. $x^2 + y^2 − 22y − 5 = 0$
83. $2x^2 + 2y^2 − 12x + 20y + 4 = 0$
84. $3x^2 + 3y^2 − 24x + 18y + 3 = 0$

In this section we looked at characteristics of equations that generated linear graphs, and graphs of parabolas and circles. Use this information and ordered pairs of your choosing to match the eight graphs given with their corresponding equation (two of the equations given have no matching graph).

a. $y = x^2 − 6x$

b. $x^2 + (y − 3)^2 = 36$

c. $x^2 + y = 9$

d. $3x − 4y = 12$

e. $y = \frac{−3}{2}x + 4$

f. $(x − 1)^2 + (y + 2)^2 = 49$

g. $(x − 3)^2 + y^2 = 16$

h. $(x − 1)^2 + (y + 2)^2 = 9$

i. $4x − 3y = 12$

j. $6x + y = x^2 + 9$
93. Spending on Cable and Satellite TV:
\[ s = 29t + 96 \]
The data from Example 1 is closely modeled by the formula shown, where \( t \) represents the year (\( t = 0 \) corresponds to the year 2000) and \( s \) represents the average amount spent per person, per year in the United States. (a) List five ordered pairs for this relation using \( t = 3, 5, 7, 9, 11 \). Does the model give a good approximation of the actual data? (b) According to the model, what will be the average amount spent on cable and satellite TV in the year 2013? (c) According to the model, in what year will annual spending surpass $500? (d) Use the table to graph this relation by hand.

94. Radius of a circumscribed circle: \[ r = \sqrt{\frac{A}{2}} \]
The radius \( r \) of a circle circumscribed around a square is found by using the formula given, where \( A \) is the area of the square. Solve the formula for \( A \) and use the result to find the area of the square shown.

95. Radar detection: A luxury liner is located at map coordinates \((5, 12)\) and has a radar system with a range of 25 nautical miles in any direction. (a) Write the equation of the circle that models the range of the ship's radar, and (b) use the distance formula to determine if the radar can pick up the liner's sister ship located at coordinates \((15, 36)\).

97. Inscribed circle: Find the equation for both the red and blue circles, then find the area of the region shaded in blue.

99. Radio broadcast range: Two radio stations may not use the same frequency if their broadcast areas overlap. Suppose station KXXRQ has a broadcast area bounded by \( x^2 + y^2 + 8x - 6y = 0 \) and WLRT has a broadcast area bounded by \( x^2 + y^2 - 10x + 4y = 0 \). Graph the circle representing each broadcast area on the same grid to determine if both stations may broadcast on the same frequency.

96. Earthquake range: The epicenter (point of origin) of a large earthquake was located at map coordinates \((3, 7)\), with the quake being felt up to 12 mi away. (a) Write the equation of the circle that models the range of the earthquake's effect. (b) Use the distance formula to determine if a person living at coordinates \((13, 1)\) would have felt the quake.

98. Inscribed triangle: The area of an equilateral triangle inscribed in a circle is given by the formula \( A = \frac{3\sqrt{3}}{4}r^2 \), where \( r \) is the radius of the circle. Find the area of the equilateral triangle shown.
100. Radio broadcast range: The emergency radio broadcast system is designed to alert the population by relaying an emergency signal to all points of the country. A signal is sent from a station whose broadcast area is bounded by \( x^2 + y^2 = 2500 \) (x and y in miles) and the signal is picked up and relayed by a transmitter with range \((x - 20)^2 + (y - 30)^2 = 900\). Graph the circle representing each broadcast area on the same grid to determine the greatest distance from the original station that this signal can be received. Be sure to scale the axes appropriately.

101. Graph the circle defined by \( x^2 + y^2 = 36 \) using a friendly window, then use the Trace feature to find the value of \( y \) when \( x = 3.6 \). Now find the value of \( y \) when \( x = 4.8 \). Explain why the values seem “interchangeable.”

102. Graph the circle defined by \((x - 3)^2 + y^2 = 16\) using a friendly window, then use the Trace feature to find the value of the \( y \)-intercepts. Show you get the same intercept by computation.

**EXTENDING THE CONCEPT**

103. Although we use the word “domain” extensively in mathematics, it is also commonly seen in literature and heard in everyday conversation. Using a college-level dictionary, look up and write out the various meanings of the word, noting how closely the definitions given are related to its mathematical use.

104. Consider the following statement, then determine whether it is true or false and discuss why. A graph will exhibit some form of symmetry if, given a point that is \( h \) units from the \( x \)-axis, \( k \) units from the \( y \)-axis, and \( d \) units from the origin, there is a second point on the graph that is a like distance from the origin and each axis.

**MAINTAINING YOUR SKILLS**

106. *(Appendix A.2)* Evaluate/Simplify the following expressions.

   a. \( \frac{x^2 x^5}{x^3} \)

   b. \( 3^3 + 3^2 + 3^1 + 3^0 + 3^{-1} \)

   c. \( 125^{-\frac{1}{3}} \)

   d. \( 27^{\frac{2}{3}} \)

   e. \( (2m^3 n)^2 \)

   f. \( (5x)^0 + 5x^0 \)

107. *(Appendix A.3)* Solve the following equation.

\[
\frac{x}{3} + \frac{1}{4} = \frac{5}{6}
\]

108. *(Appendix A.4)* Solve \( x^2 - 27 = 6x \) by factoring.

109. *(Appendix A.6)* Solve \( 1 - \sqrt{n + 3} = -n \) and check solutions by substitution. If a solution is extraneous, so state.
In preparation for sketching graphs of other equations, we'll first look more closely at the characteristics of linear graphs. While linear graphs are fairly simple models, they have many substantive and meaningful applications. For instance, most of us are aware that satellite and cable TV have been increasing in popularity since they were first introduced. A close look at Figure 1.2 from Section 1.1 reveals that spending on these forms of entertainment increased from $192 per person per year in 2003 to $281 in 2007 (Figure 1.26). From an investor's or a producer's point of view, there is a very high interest in the questions, "How fast are sales increasing? Can this relationship be modeled mathematically to help predict sales in future years?" Answers to these and other questions are precisely what our study in this section is all about.

A. The Graph of a Linear Equation

A linear equation can be identified using these three tests:

1. the exponent on any variable is one,
2. no variable occurs in a denominator, and
3. no two variables are multiplied together.

The equation \(3y = 9\) is a linear equation in one variable, while \(2x + 3y = 12\) and \(y = -\frac{3}{4}x + 4\) are linear equations in two variables. In general, we have the following definition:

**Linear Equations**

A linear equation is one that can be written in the form

\[ ax + by = c \]

where \(a, b,\) and \(c\) are real numbers, with \(a\) and \(b\) not simultaneously equal to zero.

As in Section 1.1, the most basic method for graphing a line is to simply plot a few points, then draw a straight line through the points.

**EXAMPLE 1**

**Graphing a Linear Equation in Two Variables**

Graph the equation \(3x + 2y = 4\) by plotting points.

Selecting \(x = -2, x = 0, x = 1,\) and \(x = 4\) as inputs, we compute the related outputs and enter the ordered pairs in a table. The result is

<table>
<thead>
<tr>
<th>(x) input</th>
<th>(y) output</th>
<th>((x, y)) ordered pairs</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>5</td>
<td>(-2, 5)</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>(0, 2)</td>
</tr>
<tr>
<td>1</td>
<td>1\frac{1}{2}</td>
<td>(1, 1\frac{1}{2})</td>
</tr>
<tr>
<td>4</td>
<td>-4</td>
<td>(4, -4)</td>
</tr>
</tbody>
</table>

Now try Exercises 7 through 12.
Notice that the line in Example 1 crosses the y-axis at $(0, 2)$, and this point is called the y-intercept of the line. In general, y-intercepts have the form $(0, y)$. Although difficult to see graphically, substituting $0$ for $y$ and solving for $x$ shows this line crosses the x-axis at $(\frac{3}{2}, 0)$ and this point is called the x-intercept. In general, x-intercepts have the form $(x, 0)$. The x- and y-intercepts are usually easier to calculate than other points (since $y = 0$ or $x = 0$, respectively) and we often graph linear equations using only these two points. This is called the intercept method for graphing linear equations.

**The Intercept Method**

1. Substitute 0 for $x$ and solve for $y$. This will give the y-intercept $(0, y)$.
2. Substitute 0 for $y$ and solve for $x$. This will give the x-intercept $(x, 0)$.
3. Plot the intercepts and use them to graph a straight line.

**EXAMPLE 2**

Graphing Lines Using the Intercept Method

Graph $3x + 2y = 9$ using the intercept method.

**Solution**

Substitute 0 for $x$ (y-intercept)

$$3(0) + 2y = 9$$

$$2y = 9$$

$$y = \frac{9}{2}$$

$$\left(0, \frac{9}{2}\right)$$

Substitute 0 for $y$ (x-intercept)

$$3x + 2(0) = 9$$

$$3x = 9$$

$$x = 3$$

$$\left(3, 0\right)$$

A. You've just seen how we can graph linear equations using the intercept method

**B. The Slope of a Line and Rates of Change**

After the x- and y-intercepts, we next consider the slope of a line. We see applications of this concept in many diverse areas, including the grade of a highway (trucking), the pitch of a roof (carpentry), the climb of an airplane (flying), the drainage of a field (landscaping), and the slope of a mountain (parks and recreation). While the general concept is an intuitive one, we seek to quantify the concept (assign it a numeric value) for purposes of comparison and decision making. In each of the preceding examples (grade, pitch, climb, etc.), slope is a measure of "steepness," as defined by the ratio of vertical change to horizontal change. Using a line segment through arbitrary points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, we can create the right triangle shown in Figure 1.27 to help us quantify this relationship. The figure illustrates that the vertical change or the
change in \( y \) (also called the rise) is simply the difference in \( y \)-coordinates: \( y_2 - y_1 \). The horizontal change or change in \( x \) (also called the run) is the difference in \( x \)-coordinates: \( x_2 - x_1 \). In algebra, we typically use the letter “\( m \)” to represent slope, giving \( m = \frac{y_2 - y_1}{x_2 - x_1} \) as the change in \( y \) change in \( x \). The result is called the slope formula.

### The Slope Formula
Given two points \( P_1 = (x_1, y_1) \) and \( P_2 = (x_2, y_2) \), the slope of the nonvertical line through \( P_1 \) and \( P_2 \) is

\[
m = \frac{y_2 - y_1}{x_2 - x_1}
\]

where \( x_2 \neq x_1 \).

Actually, the slope value does much more than quantify the slope of a line, it expresses a rate of change between the quantities measured along each axis. In applications of slope, the ratio change in \( y \) change in \( x \) is symbolized as \( \Delta y \Delta x \). The symbol \( \Delta \) is the Greek letter delta and has come to represent a change in some quantity, and the notation \( m = \frac{\Delta y}{\Delta x} \) is read, “slope is equal to the change in \( y \) over the change in \( x \).” Interpreting slope as a rate of change has many significant applications in college algebra and beyond.

### Example 3
Using the Slope Formula
Find the slope of the line through the given points, then use \( m = \frac{\Delta y}{\Delta x} \) to find an additional point on the line.

a. \((2, 1)\) and \((8, 4)\)

\[
m_a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{8 - 2} = \frac{3}{6} = \frac{1}{2}
\]

The slope of this line is \( \frac{1}{2} \).

b. \((-2, 6)\) and \((4, 2)\)

\[
m_b = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 6}{4 - (-2)} = \frac{-4}{6} = -\frac{2}{3}
\]

The slope of this line is \( -\frac{2}{3} \).

Using \( \frac{\Delta y}{\Delta x} = \frac{1}{2} \), we note that \( y \) increases 1 unit (the \( y \)-value is positive), as \( x \) increases 2 units. Since \((8, 4)\) is known to be on the line, the point \((8 + 2, 4 + 1) = (10, 5)\) must also be on the line.

Using \( \frac{\Delta y}{\Delta x} = -\frac{2}{3} \), we note that \( y \) decreases 2 units (the \( y \)-value is negative), as \( x \) increases 3 units. Since \((4, 2)\) is known to be on the line, the point \((4 + 3, 2 - 2) = (7, 0)\) must also be on the line.

Now try Exercises 33 through 40
When using the slope formula, try to avoid these common errors.

1. The order that the \( x \)- and \( y \)-coordinates are subtracted must be consistent, since \( \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \).

2. The vertical change (involving the \( y \)-values) always occurs in the numerator:
   \( \frac{y_2 - y_1}{x_2 - x_1} \neq \frac{y_1 - y_2}{x_1 - x_2} \).

3. When \( x_1 \) or \( y_1 \) is negative, use parentheses when substituting into the formula to prevent confusing the negative sign with the subtraction operation.

**Example 4**

**Interpreting the Slope Formula as a Rate of Change**

Jimmy works on the assembly line for an auto parts remanufacturing company. By 9:00 A.M., his group has assembled 29 carburetors. By 12:00 noon, they have completed 87 carburetors. Assuming the relationship is linear, find the slope of the line and discuss its meaning in this context.

First write the information as ordered pairs using \( c \) to represent the carburetors assembled and \( t \) to represent time. This gives \((t_1, c_1) = (9, 29)\) and \((t_2, c_2) = (12, 87)\). The slope formula then gives:

\[
\frac{\Delta c}{\Delta t} = \frac{c_2 - c_1}{t_2 - t_1} = \frac{87 - 29}{12 - 9} = \frac{58}{3} \text{ or } 19\frac{1}{3}
\]

Here the slope ratio measures \( \frac{\text{carburetors assembled}}{\text{hours}} \), and we see that Jimmy’s group can assemble 58 carburetors every 3 hr, or about 19\frac{1}{3} carburetors per hour.

**Positive and Negative Slope**

If you’ve ever traveled by air, you’ve likely heard the announcement, “Ladies and gentlemen, please return to your seats and fasten your seat belts as we begin our descent.” For a time, the descent of the airplane follows a linear path, but the slope of the line is negative since the altitude of the plane is decreasing. Positive and negative slopes, as well as the rate of change they represent, are important characteristics of linear graphs. In Example 3(a), the slope was a positive number \((m > 0)\) and the line will slope upward from left to right since the \( y \)-values are increasing. If \( m < 0 \) as in Example 3(b), the slope of the line is negative and the line slopes downward as you move left to right since \( y \)-values are decreasing.
Example 5

Applying Slope as a Rate of Change in Altitude

At a horizontal distance of 10 mi after take-off, an airline pilot receives instructions to decrease altitude from their current level of 20,000 ft. A short time later, they are 17.5 mi from the airport at an altitude of 10,000 ft. Find the slope ratio for the descent of the plane and discuss its meaning in this context. Recall that 1 mi = 5280 ft.

Solution

Let \( a \) represent the altitude of the plane and \( d \) its horizontal distance from the airport. Converting all measures to feet, we have \((d_1, a_1) = (52,800, 20,000)\) and \((d_2, a_2) = (92,400, 10,000)\), giving

\[
\frac{\Delta a}{\Delta d} = \frac{a_2 - a_1}{d_2 - d_1} = \frac{10,000 - 20,000}{92,400 - 52,800} = \frac{-10,000}{39,600} = -\frac{25}{99}
\]

Since this slope ratio measures \(\frac{\text{Change in Altitude}}{\text{Change in Distance}}\), we note the plane is decreasing 25 ft in altitude for every 99 ft it travels horizontally.

C. Horizontal Lines and Vertical Lines

Horizontal and vertical lines have a number of important applications, from finding the boundaries of a given graph (the domain and range), to performing certain tests on nonlinear graphs. To better understand them, consider that in one dimension, the graph of \( x = 2 \) is a single point (Figure 1.28), indicating a location on the number line 2 units from zero in the positive direction. In two dimensions, the equation \( x = 2 \) represents all points with an \( x \)-coordinate of 2. A few of these are graphed in Figure 1.29, but since there are an infinite number, we end up with a solid vertical line whose equation is \( x = 2 \) (Figure 1.30).

The same idea can be applied to horizontal lines. In two dimensions, the equation \( y = 4 \) represents all points with a \( y \)-coordinate of positive 4, and there are an infinite number of these as well. The result is a solid horizontal line whose equation is \( y = 4 \). See Exercises 49 through 54.

Worthy of Note

If we write the equation \( x = 2 \) in the form \( ax + by = c \), the equation becomes \( x + 0y = 2 \), since the original equation has no \( y \)-variable. Notice that regardless of the value chosen for \( y \), \( x \) will always be 2 and we end up with the set of ordered pairs \((2, y)\), which gives us a vertical line.
So far, the slope formula has only been applied to lines that were nonhorizontal or nonvertical. So what is the slope of a horizontal line? On an intuitive level, we expect that a perfectly level highway would have an incline or slope of zero. In general, for any two points on a horizontal line, \( y_2 = y_1 \) and \( y_2 - y_1 = 0 \), giving a slope of \( m = \frac{y_2 - y_1}{x_2 - x_1} = 0 \). For any two points on a vertical line, \( x_2 = x_1 \) and \( x_2 - x_1 = 0 \), making the slope ratio undefined: \( m = \frac{y_2 - y_1}{0} \) (see Figures 1.31 and 1.32).

**Figure 1.31**
For any horizontal line, \( y_2 = y_1 \)

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0
\]

**Figure 1.32**
For any vertical line, \( x_2 = x_1 \)

\[
\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{x_1 - x_1} = \frac{0}{0} = \text{undefined}
\]

**The Slope of a Horizontal Line**
The slope of any horizontal line is zero.

**The Slope of a Vertical Line**
The slope of any vertical line is undefined.

---

**EXAMPLE 6**

Calculating Slopes

The federal minimum wage remained constant from 1997 through 2006. However, the buying power (in 1996 dollars) of these wage earners fell each year due to inflation (see Table 1.3). This decrease in buying power is approximated by the red line shown.

a. Using the data or graph, find the slope of the line segment representing the minimum wage.

b. Select two points on the line representing buying power to approximate the slope of the line segment, and explain what it means in this context.

**Table 1.3**

<table>
<thead>
<tr>
<th>Time ( t ) (years)</th>
<th>Minimum wage ( w )</th>
<th>Buying power ( p )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>5.15</td>
<td>5.03</td>
</tr>
<tr>
<td>1998</td>
<td>5.15</td>
<td>4.96</td>
</tr>
<tr>
<td>1999</td>
<td>5.15</td>
<td>4.85</td>
</tr>
<tr>
<td>2000</td>
<td>5.15</td>
<td>4.69</td>
</tr>
<tr>
<td>2001</td>
<td>5.15</td>
<td>4.56</td>
</tr>
<tr>
<td>2002</td>
<td>5.15</td>
<td>4.49</td>
</tr>
<tr>
<td>2003</td>
<td>5.15</td>
<td>4.39</td>
</tr>
<tr>
<td>2004</td>
<td>5.15</td>
<td>4.28</td>
</tr>
<tr>
<td>2005</td>
<td>5.15</td>
<td>4.14</td>
</tr>
<tr>
<td>2006</td>
<td>5.15</td>
<td>4.04</td>
</tr>
</tbody>
</table>
Solution

WORTHY OF NOTE
In the context of lines, try to avoid saying that a horizontal line has "no slope," since it's unclear whether a slope of zero or an undefined slope is intended.

C. You've just seen how we can graph horizontal and vertical lines

Now try Exercises 55 and 56

D. Parallel and Perpendicular Lines

Two lines in the same plane that never intersect are called parallel lines. When we place these lines on the coordinate grid, we find that "never intersect" is equivalent to saying "the lines have equal slopes but different y-intercepts." In Figure 1.33, notice the rise and run of each line is identical, and that by counting $\frac{\Delta y}{\Delta x}$ both lines have slope $m = \frac{3}{4}$.

Figure 1.33

Parallel Lines

Given $L_1$ and $L_2$ are distinct, nonvertical lines with slopes of $m_1$ and $m_2$, respectively.

1. If $m_1 = m_2$, then $L_1$ is parallel to $L_2$.
2. If $L_1$ is parallel to $L_2$, then $m_1 = m_2$.

In symbols, we write $L_1 \parallel L_2$.

Any two vertical lines (undefined slope) are parallel.

EXAMPLE 7A Determine Whether Two Lines Are Parallel

Teladango Park has been mapped out on a rectangular coordinate system, with a ranger station at $(0, 0)$. Brendan and Kapi are at coordinates $(-24, -18)$ and have set a direct course for the pond at $(11, 10)$. Caden and Kymani are at $(-27, 1)$ and are heading straight to the lookout tower at $(-2, 21)$. Are they hiking on parallel or nonparallel courses?
Solution  
To respond, we compute the slope of each trek across the park.

For Brendan and Kapi:  
For Caden and Kymani:

\[
\begin{align*}
\frac{y_2 - y_1}{x_2 - x_1} &= \frac{10 - (-18)}{11 - (-24)} \quad & \frac{y_2 - y_1}{x_2 - x_1} &= \frac{21 - 1}{-2 - (-27)} \\
&= \frac{28}{35} \quad & &= \frac{20}{25} \\
&= \frac{4}{5} \quad & &= \frac{4}{5}
\end{align*}
\]

Since the slopes are equal, the two groups are hiking on parallel courses.

Two lines in the same plane that intersect at right angles are called **perpendicular lines**. Using the coordinate grid, we note that **intersect at right angles** suggests that **their slopes are negative reciprocals**. While certainly not a proof, notice in Figure 1.34, the ratio **rise** for L₁ is \(\frac{3}{4}\) and the ratio **run** for L₂ is \(\frac{-3}{4}\). Alternatively, we can say their slopes **have a product of** \(-1\), since \(m_1 \cdot m_2 = -1\) implies \(m_1 = -\frac{1}{m_2}\).

**Figure 1.34**  
Generic plane  
Coordinate plane

**Perpendicular Lines**

Given \(L_1\) and \(L_2\) are distinct, nonvertical lines with slopes of \(m_1\) and \(m_2\), respectively.

1. If \(m_1 \cdot m_2 = -1\), then \(L_1\) is perpendicular to \(L_2\).
2. If \(L_1\) is perpendicular to \(L_2\), then \(m_1 \cdot m_2 = -1\).

In symbols we write \(L_1 \perp L_2\).  
**Any vertical line (undefined slope) is perpendicular to any horizontal line (slope \(m = 0\)).**

We can easily find the slope of a line perpendicular to a second line whose slope is known or can be found—just find the reciprocal and make it negative. For a line with slope \(m_1 = -\frac{3}{4}\), any line perpendicular to it will have a slope of \(m_2 = \frac{4}{3}\). For \(m_1 = -5\), the slope of any line perpendicular would be \(m_2 = \frac{1}{5}\).

**Example 7B**  
**Determining Whether Two Lines Are Perpendicular**

The three points \(P_1 = (5, 1)\), \(P_2 = (3, -2)\), and \(P_3 = (-3, 2)\) form the vertices of a triangle. Use these points to draw the triangle, then use the slope formula to determine if they form a **right** triangle.
Solution

For a right triangle to be formed, two of the lines through these points must be perpendicular (forming a right angle). From Figure 1.35, it appears a right triangle is formed, but we must verify that two of the sides are actually perpendicular. Using the slope formula, we have:

For $P_1$ and $P_2$

$$m_1 = \frac{-2 - 1}{3 - 5} = \frac{-3}{-2} = \frac{3}{2}$$

For $P_1$ and $P_3$

$$m_2 = \frac{2 - 1}{-3 - 5} = \frac{1}{-8} = -\frac{1}{8}$$

For $P_2$ and $P_3$

$$m_3 = \frac{2 - (-2)}{-3 - 3} = \frac{4}{-6} = -\frac{2}{3}$$

Since $m_1 \cdot m_3 = -1$, the triangle has a right angle and must be a right triangle.

D. You've just seen how we can identify parallel and perpendicular lines

E. Applications of Linear Equations

The graph of a linear equation can be used to help solve many applied problems. If the numbers you're working with are either very small or very large, scale the axes appropriately. This can be done by letting each tic mark represent a smaller or larger unit so the data points given will fit on the grid. Also, many applications use only nonnegative values and although points with negative coordinates may be used to graph a line, only ordered pairs in QI can be meaningfully interpreted.

EXAMPLE 8

Applying a Linear Equation Model—Commission Sales

Use the information given to create a linear equation model in two variables, then graph the line and answer the question posed:

A salesperson gets a daily $20 meal allowance plus $7.50 for every item she sells. How many sales are needed for a daily income of $125?

Verify your answer by graphing the line on a calculator and using the Trace feature.

Algebraic Solution

verbal model: Daily income ($y$) equals $7.5$ per sale ($x$) + $20$

equation model: $y = 7.5x + 20$

Using $x = 0$ and $x = 10$, we find $(0, 20)$ and $(10, 95)$ are points on this line and these are used to sketch the graph. From the graph, it appears that 14 sales are needed to generate a daily income of $125.00.

Since daily income is given as $125, we substitute 125 for $y$ and solve for $x$.

$125 = 7.5x + 20$

$105 = 7.5x$

$14 = x$

substitute 125 for $y$

subtract 20

divide by 7.5
Graphical Solution

Begin by entering the equation \( y = 7.5x + 20 \) on the graphing calculator screen, recognizing that in this context, both the input and output values must be positive. Reasoning that 10 sales will net $95 (less than $125) and 20 sales will net $170 (more than $125), we set the viewing window as shown in Figure 1.36. We can then use the trace feature to estimate the number of sales needed. The result shows that income is close to $125 when \( x \) is close to 14 (Figure 1.37). In addition to letting us trace along a graph, the trace option enables us to evaluate the equation at specific points. Simply entering the number “14” causes the calculator to accept 14 as the desired input (Figure 1.38), and after pressing enter, it verifies that (14, 125) is indeed a point on the graph (Figure 1.39).

---

E. You’ve just seen how we can apply linear equations in context.

---

1.2 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. To find the \( x \)-intercept of a line, substitute ______ for \( y \) and solve for \( x \). To find the \( y \)-intercept, substitute ______ for \( x \) and solve for \( y \).

2. The slope formula is \( m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \), and indicates a rate of change between the \( x \)- and \( y \)-variables.

3. If \( m < 0 \), the slope of the line is ______ and the line slopes ______ from left to right.

4. The slope of a horizontal line is ______, the slope of a vertical line is ______, and the slopes of two parallel lines are ______.

5. Discuss/Explain If \( m_1 = 2.1 \) and \( m_2 = 2.01 \), will the lines intersect? If \( m_1 = \frac{2}{3} \) and \( m_2 = -\frac{2}{3} \), are the lines perpendicular?

6. Discuss/Explain the relationship between the slope formula, the Pythagorean theorem, and the distance formula. Include several illustrations.
DEVELOPING YOUR SKILLS

Create a table of values for each equation and sketch the graph.

7. \(2x + 3y = 6\)  
8. \(-3x + 5y = 10\)

\[
\begin{array}{c|c}
 x & y \\
\hline
 & \\
\end{array}
\quad
\begin{array}{c|c}
 x & y \\
\hline
 & \\
\end{array}
\]

9. \(y = \frac{3}{2}x + 4\)  
10. \(y = \frac{5}{3}x - 3\)

\[
\begin{array}{c|c}
 x & y \\
\hline
 & \\
\end{array}
\quad
\begin{array}{c|c}
 x & y \\
\hline
 & \\
\end{array}
\]

11. If you completed Exercise 9, verify that \((-3, -0.5)\) and \((\frac{3}{2}, \frac{19}{4})\) also satisfy the equation given. Do these points appear to be on the graph you sketched?

12. If you completed Exercise 10, verify that \((-1.5, -5.5)\) and \((\frac{11}{3}, \frac{27}{6})\) also satisfy the equation given. Do these points appear to be on the graph you sketched?

Graph the following equations using the intercept method. Plot a third point as a check.

13. \(3x + y = 6\)  
14. \(-2x + y = 12\)

15. \(5y - x = 5\)  
16. \(-4y + x = 8\)

17. \(-5x + 2y = 6\)  
18. \(3y + 4x = 9\)

19. \(2x - 5y = 4\)  
20. \(-6x + 4y = 8\)

21. \(2x + 3y = -12\)  
22. \(-3x - 2y = 6\)

23. \(y = \frac{1}{2}x\)  
24. \(y = \frac{2}{3}x\)

25. \(y - 25 = 50x\)  
26. \(y + 30 = 60x\)

27. \(y = -\frac{2}{5}x - 2\)  
28. \(y = \frac{3}{4}x + 2\)

29. \(2y - 3x = 0\)  
30. \(y + 3x = 0\)

31. \(3y + 4x = 12\)  
32. \(-2x + 5y = 8\)

Compute the slope of the line through the given points, then graph the line and use \(m = \frac{\Delta y}{\Delta x}\) to find two additional points on the line. Answers may vary.

33. \((3, 5), (4, 6)\)  
34. \((-2, 3), (5, 8)\)

35. \((10, 3), (4, -5)\)  
36. \((-3, -1), (0, 7)\)

37. \((1, -8), (-3, 7)\)  
38. \((-5, 5), (0, -5)\)

39. \((-3, 6), (4, 2)\)  
40. \((-2, -4), (-3, -1)\)

41. The graph shown models the relationship between the cost of a new home and the size of the home in square feet. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate the cost of a 3000 ft\(^2\) home.

42. The graph shown models the relationship between the volume of garbage that is dumped in a landfill and the number of commercial garbage trucks that enter the site. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate the number of trucks entering the site daily if 1000 m\(^3\) of garbage is dumped per day.

43. The graph shown models the relationship between the distance of an aircraft carrier from its home port and the number of hours since departure. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate the distance from port after 8.25 hours.

44. The graph shown models the relationship between the number of circuit boards that have been assembled at a factory and the number of hours since starting time. (a) Determine the slope of the line and interpret what the slope ratio means in this context and (b) estimate how many hours the factory has been running if 225 circuit boards have been assembled.
45. **Height and weight:** While there are many exceptions, numerous studies have shown a close relationship between an average height and average weight. Suppose a person 70 in. tall weighs 165 lb, while a person 64 in. tall weighs 142 lb. Assuming the relationship is linear, (a) find the slope of the line and discuss its meaning in this context and (b) determine how many pounds are added for each inch of height.

46. **Rate of climb:** Shortly after takeoff, a plane increases altitude at a constant (linear) rate. In 5 min the altitude is 10,000 ft. Fifteen minutes after takeoff, the plane has reached its cruising altitude of 32,000 ft. (a) Find the slope of the line and discuss its meaning in this context and (b) determine how long it takes the plane to climb from 12,200 ft to 25,400 ft.

47. **Sewer line slope:** Fascinated at how quickly the plumber was working, Ryan watched with great interest as the new sewer line was laid from the house to the main line, a distance of 48 ft. At the edge of the house, the sewer line was 6 in. underground. If the plumber tied in to the main line at a depth of 18 in., what is the slope of the (sewer) line? What does this slope indicate?

48. **Slope (pitch) of a roof:** A contractor goes to a lumber yard to purchase some trusses (the triangular frames) for the roof of a house. Many sizes are available, so the contractor takes some measurements to ensure the roof will have the desired slope. In one case, the height of the truss (base to ridge) was 4 ft, with a width of 24 ft (eave to eave). Find the slope of the roof if these trusses are used. What does this slope indicate?

Graph each line using two or three ordered pairs that satisfy the equation.

49. \( x = -3 \)

50. \( y = 4 \)

51. \( x = 2 \)

52. \( y = -2 \)

Write the equation for each line \(L_1\) and \(L_2\) shown. Specifically state their point of intersection.

53. [Diagram of \(L_1\) and \(L_2\) with points labeled]

54. [Diagram of \(L_1\) and \(L_2\) with points labeled]

55. **Supreme Court justices:** The table given shows the total number of justices \(j\) sitting on the Supreme Court of the United States for selected time periods \(t\) (in decades), along with the number of nonmale, nonwhite justices \(n\) for the same years. (a) Use the data to graph the linear relationship between \(t\) and \(j\), then determine the slope of the line and discuss its meaning in this context. (b) Use the data to graph the linear relationship between \(t\) and \(n\), then determine the slope of the line and discuss its meaning.

<table>
<thead>
<tr>
<th>Time (t) (1960 (\rightarrow) 0)</th>
<th>Justices (j)</th>
<th>Nonwhite, nonmale (n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>1</td>
</tr>
<tr>
<td>20</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>30</td>
<td>9</td>
<td>3</td>
</tr>
<tr>
<td>40</td>
<td>9</td>
<td>4</td>
</tr>
<tr>
<td>50</td>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>

56. **Boiling temperature:** The table shown gives the boiling temperature \(t\) of water as related to the altitude \(h\). Use the data to graph the linear relationship between \(h\) and \(t\), then determine the slope of the line and discuss its meaning in this context.

<table>
<thead>
<tr>
<th>Altitude (h) (ft)</th>
<th>Boiling Temperature (t) (°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>212.0</td>
</tr>
<tr>
<td>1000</td>
<td>210.2</td>
</tr>
<tr>
<td>2000</td>
<td>208.4</td>
</tr>
<tr>
<td>3000</td>
<td>206.6</td>
</tr>
<tr>
<td>4000</td>
<td>204.8</td>
</tr>
<tr>
<td>5000</td>
<td>203.0</td>
</tr>
<tr>
<td>6000</td>
<td>201.2</td>
</tr>
</tbody>
</table>

Two points on \(L_1\) and two points on \(L_2\) are given. Use the slope formula to determine if lines \(L_1\) and \(L_2\) are parallel, perpendicular, or neither.

57. \(L_1:\ (-2, 0)\) and \((0, 6)\) \hspace{0.5cm} \(L_2:\ (1, 8)\) and \((0, 5)\)

58. \(L_1:\ (1, 10)\) and \((-1, 7)\) \hspace{0.5cm} \(L_2:\ (0, 3)\) and \((1, 5)\)

59. \(L_1:\ (-3, -4)\) and \((0, 1)\) \hspace{0.5cm} \(L_2:\ (0, 0)\) and \((-4, 4)\)

60. \(L_1:\ (6, 2)\) and \((8, -2)\) \hspace{0.5cm} \(L_2:\ (5, 1)\) and \((3, 0)\)

61. \(L_1:\ (6, 3)\) and \((8, 7)\) \hspace{0.5cm} \(L_2:\ (7, 2)\) and \((6, 0)\)

62. \(L_1:\ (-5, -1)\) and \((4, 4)\) \hspace{0.5cm} \(L_2:\ (4, -7)\) and \((8, 10)\)
In Exercises 63 to 68, three points that form the vertices of a triangle are given. Use the points to draw the triangle, then use the slope formula to determine if any of the triangles are right triangles. Also see Exercises 43–48 in Section 1.1.

63. (−3, 7), (2, 2), (5, 5)  
64. (7, 0), (−1, 0), (7, 4)  
65. (−4, 3), (−7, −1), (3, −2)  
66. (5, 2), (0, −3), (4, −4)  
67. (−3, 2), (−1, 5), (−6, 4)  
68. (0, 0), (−5, 2), (2, −5)

**WORKING WITH FORMULAS**

69. Human life expectancy: \( L = 0.15T + 73.7 \)

In the United States, the average life expectancy has been steadily increasing over the years due to better living conditions and improved medical care. This relationship is modeled by the formula shown, where \( L \) is the average life expectancy and \( T \) is number of years since 1980. (a) What was the life expectancy in the year 2010? (b) In what year will average life expectancy reach 79 yr?

70. Interest earnings: \( 100I = 35,000T \)

If $5000 dollars is invested in an account paying 7% simple interest, the amount of interest earned is given by the formula shown, where \( I \) is the interest and \( T \) is the time in years. Begin by solving the formula for \( I \). (a) How much interest is earned in 5 yr? (b) How much is earned in 10 yr? (c) Use the two points (5 yr, interest) and (10 yr, interest) to calculate the slope of this line. What do you notice?

**APPLICATIONS**

71. Business depreciation: A business purchases a copier for $8500 and anticipates it will depreciate in value $1250 per year.

   a. What is the copier’s value after 4 yr of use?
   b. How many years will it take for this copier’s value to decrease to $2250?

72. Baseball card value: After purchasing an autographed baseball card for $85, its value increases by $1.50 per year.

   a. What is the card’s value 7 yr after purchase?
   b. How many years will it take for this card’s value to reach $100?

73. Water level: During a long drought, the water level in a local lake decreased at a rate of 3 in. per month. The water level before the drought was 300 in.

   a. What was the water level after 9 months of drought?
   b. How many months will it take for the water level to decrease to 20 ft?

74. Gas mileage: When empty, a large dump-truck gets about 15 mi per gallon. It is estimated that for each 3 tons of cargo it hauls, gas mileage decreases by 2 mi per gallon.

   a. If 10 tons of cargo is being carried, what is the truck’s mileage?
   b. If the truck’s mileage is down to 10 mi per gallon, how much weight is it carrying?

75. Parallel/nonparallel roads: Aberville is 38 mi north and 12 mi west of Boschertown, with a straight “farm and machinery” road (FM 1960) connecting the two cities. In the next county, Crownsburg is 30 mi north and 9.5 mi west of Dower, and these cities are likewise connected by a straight road (FM 830). If the two roads continued indefinitely in both directions, would they intersect at some point?

76. Perpendicular/nonperpendicular course headings: Two shrimp trawlers depart Charleston Harbor at the same time. One heads for the shrimp prelims located 12 mi north and 3 mi east of the harbor. The other heads for a point 2 mi south and 8 mi east of the harbor. Assuming the harbor is at (0, 0), are the routes of the trawlers perpendicular? If so, how far apart are the boats when they reach their destinations (to the nearest one-tenth mi)?
77. **Cost of college:** For the years 2000 to 2008, the cost of tuition and fees per semester (in constant dollars) at a public 4-yr college can be approximated by the equation \( y = 386x + 3500 \), where \( y \) represents the cost in dollars and \( x = 0 \) represents the year 2000. Use the equation to find: (a) the cost of tuition and fees in 2010 and (b) the year this cost will exceed $9000.

*Source: The College Board*

78. **Female physicians:** In 1960 only about 7% of physicians were female. Soon after, this percentage began to grow dramatically. For the years 1990 to 2000, the percentage of physicians that were female can be approximated by the equation \( y = 0.6x + 18.1 \), where \( y \) represents the percentage (as a whole number) and \( x = 0 \) represents the year 1990. Use the equation to find: (a) the percentage of physicians that were female in 2000 and (b) the projected year this percentage would have exceeded 30%.

*Source: American Journal of Public Health*

**EXTENDING THE CONCEPT**

81. If the lines \( 4y + 2x = -5 \) and \( 3y + ax = -2 \) are perpendicular, what is the value of \( a \)?

82. Let \( m_1, m_2, m_3, \) and \( m_4 \) be the slopes of lines \( L_1, L_2, L_3, \) and \( L_4 \), respectively. Which of the following statements is true?
   a. \( m_4 < m_1 < m_3 < m_2 \)
   b. \( m_3 < m_2 < m_4 < m_1 \)
   c. \( m_3 < m_4 < m_3 < m_1 \)
   d. \( m_1 < m_3 < m_4 < m_2 \)
   e. \( m_1 < m_4 < m_3 < m_2 \)

**MAINTAINING YOUR SKILLS**

84. *(1.1)* Name the center and radius of the circle defined by \((x - 3)^2 + (y + 4)^2 = 169\)

85. *(Appendix A.6)* Compute the sum and product indicated:
   a. \( \sqrt{20} + 3\sqrt{45} - \sqrt{5} \)
   b. \( (3 + \sqrt{5})(3 - \sqrt{5}) \)

86. *(Appendix A.4)* Solve the equation by factoring, then check the result(s) using substitution:
   \( 12x^2 - 44x - 45 = 0 \)

87. *(Appendix A.5)* Factor the following polynomials completely:
   a. \( x^3 - 3x^2 - 4x + 12 \)
   b. \( x^2 - 23x - 24 \)
   c. \( x^2 - 125 \)

89. **Decrease in smokers:** For the years 1990 to 2000, the percentage of the U.S. adult population who were smokers can be approximated by the equation \( y = \frac{-13}{22}x + 28.7 \), where \( y \) represents the percentage of smokers (as a whole number) and \( x = 0 \) represents 1990. Use the equation to find: (a) the percentage of adults who smoked in the year 2005 and (b) the year the percentage of smokers is projected to fall below 15%.

*Source: WebMD*

90. **Temperature and cricket chirps:** Biologists have found a strong relationship between temperature and the number of times a cricket chirps. This is modeled by the equation \( T = \frac{1}{3}N + 40 \), where \( N \) is the number of times the cricket chirps per minute and \( T \) is the temperature in Fahrenheit. Use the equation to find: (a) the outdoor temperature if the cricket is chirping 48 times per minute and (b) the number of times a cricket chirps if the temperature is 70°.

*a*
1.3 Functions, Function Notation, and the Graph of a Function

LEARNING OBJECTIVES

In Section 1.3 you will see how we can:

- A. Distinguish the graph of a function from that of a relation
- B. Determine the domain and range of a function
- C. Use function notation and evaluate functions
- D. Read and interpret information given graphically

In this section we introduce one of the most central ideas in mathematics—the concept of a function. Functions can model the cause-and-effect relationship that is so important to using mathematics as a decision-making tool. In addition, the study will help to unify and expand on many ideas that are already familiar.

A. Functions and Relations

There is a special type of relation that merits further attention. A function is a relation where each element of the domain corresponds to exactly one element of the range. In other words, for each first coordinate or input value, there is only one possible second coordinate or output.

**Functions**

A function is a relation that pairs each element from the domain with exactly one element from the range.

If the relation is defined by a mapping, we need only check that each element of the domain is mapped to exactly one element of the range. This is indeed the case for the mapping $P \rightarrow B$ from Figure 1.1 (page 2), where we saw that each person corresponded to only one birthday, and that it was impossible for one person to be born on two different days. For the relation $x = |y|$ shown in Figure 1.6 (page 4), each element of the domain except zero is paired with more than one element of the range. The relation $x = |y|$ is not a function.

**EXAMPLE 1**

Determining Whether a Relation is a Function

Three different relations are given in mapping notation below. Determine whether each relation is a function.

<table>
<thead>
<tr>
<th>a. Person</th>
<th>Room</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marie</td>
<td>270</td>
</tr>
<tr>
<td>Pesky</td>
<td>268</td>
</tr>
<tr>
<td>Bo</td>
<td>274</td>
</tr>
<tr>
<td>Johnny</td>
<td>276</td>
</tr>
<tr>
<td>Rick</td>
<td>272</td>
</tr>
<tr>
<td>Annie</td>
<td>282</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>b. Pet</th>
<th>Weight (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fido</td>
<td>450</td>
</tr>
<tr>
<td>Bossy</td>
<td>550</td>
</tr>
<tr>
<td>Silver</td>
<td>2</td>
</tr>
<tr>
<td>Frisky</td>
<td>40</td>
</tr>
<tr>
<td>Polly</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>c. War</th>
<th>Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Civil War</td>
<td>1963</td>
</tr>
<tr>
<td>World War I</td>
<td>1950</td>
</tr>
<tr>
<td>World War II</td>
<td>1939</td>
</tr>
<tr>
<td>Korean War</td>
<td>1917</td>
</tr>
<tr>
<td>Vietnam War</td>
<td>1861</td>
</tr>
</tbody>
</table>

**Solution**

Relation (a) is a function, since each person corresponds to exactly one room. This relation pairs math professors with their respective office numbers. Notice that while two people can be in one office, it is impossible for one person to physically be in two different offices.

Relation (b) is not a function, since we cannot tell whether Polly the Parrot weighs 2 lb or 3 lb (one element of the domain is mapped to two elements of the range).

Relation (c) is a function, where each major war is paired with the year it began.

Now try Exercises 7 through 10.
If the relation is pointwise-defined or given as a set of individual and distinct plotted points, we need only check that no two points have the same first coordinate with a different second coordinate. This gives rise to an alternative definition for a function.

**Functions (Alternate Definition)**

A function is a set of ordered pairs \((x, y)\), in which each first component is paired with only one second component.

### EXAMPLE 2 > Identifying Functions

Two relations named \(f\) and \(g\) are given: \(f\) is pointwise-defined (stated as a set of ordered pairs), while \(g\) is given as a set of plotted points. Determine whether each is a function.

\[
\begin{align*}
f: & \quad (-3, 0), (1, 4), (2, -5), (4, 2), (-3, -2), (3, 6), (0, -1), (4, -5), \text{ and } (6, 1) \\
\end{align*}
\]

**Solution**

The relation \(f\) is not a function, since \(-3\) is paired with two different outputs: \((-3, 0)\) and \((-3, -2)\).

The relation \(g\) shown in the figure is a function. Each input corresponds to exactly one output, otherwise one point would be directly above the other and have the same first coordinate.

Now try Exercises 11 through 18

The graphs of \(y = x - 1\) and \(x = |y|\) from Section 1.1 offer additional insight into the definition of a function. Figure 1.40 shows the line \(y = x - 1\) with emphasis on the plotted points \((4, 3)\) and \((-3, -4)\). The vertical movement shown from the \(x\)-axis to a point on the graph illustrates the pairing of a given \(x\)-value with one related \(y\)-value. Note the vertical line shows only one related \(y\)-value \((x = 4\) is paired with only \(y = 3\)). Figure 1.41 gives the graph of \(x = |y|\), highlighting the points \((4, 4)\) and \((4, -4)\). The vertical movement shown here branches in two directions, associating one \(x\)-value with more than one \(y\)-value. This shows the relation \(y = x - 1\) is also a function, while the relation \(x = |y|\) is not.

![Figure 1.40](image1.png)

![Figure 1.41](image2.png)

This "vertical connection" of a location on the \(x\)-axis to a point on the graph can be generalized into a vertical line test for functions.
**Vertical Line Test**

A given graph is the graph of a function, if and only if every vertical line intersects the graph in at most one point.

Applying the test to the graph in Figure 1.40 helps to illustrate that the graph of any nonvertical line must be the graph of a function, as is the graph of any pointwise-defined relation where no x-coordinate is repeated. Compare the relations \( f \) and \( g \) from Example 2.

**EXAMPLE 3**

**Using the Vertical Line Test**

Use the vertical line test to determine if any of the relations shown (from Section 1.1) are functions.

**Solution**

Visualize a vertical line on each coordinate grid (shown in solid blue), then mentally shift the line to the left and right as shown in Figures 1.42, 1.43, and 1.44 (dashed lines). In Figures 1.42 and 1.43, every vertical line intersects the graph only once, indicating both \( y = x^2 - 2x \) and \( y = \sqrt{9 - x^2} \) are functions. In Figure 1.44, a vertical line intersects the graph twice for any \( x > 0 \) [for instance, both \((4, 2)\) and \((4, -2)\) are on the graph]. The relation \( x = y^2 \) is not a function.

**WORTHY OF NOTE**

For relations and functions, a good way to view the distinction is to consider a mail carrier. It is possible for the carrier to put more than one letter into the same mailbox (more than one \( x \) going to the same \( y \)), but quite impossible for the carrier to place the same letter in two different boxes (one \( x \) going to two \( y \)’s).

**EXAMPLE 4**

**Using the Vertical Line Test**

Use a table of values to graph the relations defined by

\[
\begin{align*}
a. \quad y &= |x| \\
b. \quad y &= \sqrt{x}
\end{align*}
\]

then use the vertical line test to determine whether each relation is a function.

**Solution**

a. For \( y = |x| \), using input values from \( x = -4 \) to \( x = 4 \) produces the following table and graph (Figure 1.45). Note the result is a V-shaped graph that "opens upward." The point \((0, 0)\) of this absolute value graph is called the **vertex**.

Since any vertical line will intersect the graph in at most one point, this is the graph of a function.
b. For \( y = \sqrt{x} \), values less than zero do not produce a real number, so our graph actually begins at \((0, 0)\) (see Figure 1.46). Completing the table for nonnegative values produces the graph shown, which appears to rise to the right and remains in the first quadrant. Since any vertical line will intersect this graph in at most one place, \( y = \sqrt{x} \) is also a function.

---

**B. The Domain and Range of a Function**

**Vertical Boundary Lines and the Domain**

In addition to its use as a graphical test for functions, a vertical line can help determine the domain of a function from its graph. For the graph of \( y = \sqrt{x} \) (Figure 1.46), a vertical line will not intersect the graph until \( x = 0 \), and then will intersect the graph for all values \( x \geq 0 \) (showing the function is defined for these values). These **vertical boundary lines** indicate the domain is \( x \geq 0 \).

Instead of using a simple inequality to write the domain and range, we will often use (1) a form of **set notation**, (2) a **number line** graph, or (3) **interval notation**. Interval notation is a symbolic way of indicating a selected interval of the real numbers. When a number acts as the **boundary point** for an interval (also called an **endpoint**), we use a left bracket \([\) or a right bracket \(])\) to indicate **inclusion** of the endpoint. If the boundary point is **not included**, we use a left parenthesis \((-\) or right parenthesis \(\)\).

---

**WORTHY OF NOTE**

On a number line, some texts will use an open dot "••" to mark the location of an endpoint that is not included, and a closed dot "••" for an included endpoint.
EXAMPLE 5  ➤ Using Notation to State the Domain and Range

Model the given phrase using the correct inequality symbol. Then state the result in set notation, graphically, and in interval notation: “The set of real numbers greater than or equal to 1.”

Solution ➤

Let \( n \) represent the number: \( n \geq 1 \).

- Set notation: \( \{n | n \geq 1\} \)
- Graph: [Graph with a horizontal line at \( n \geq 1 \) extending to the right]
- Interval notation: \( n \in [1, \infty) \)

Now try Exercises 35 through 50 ➤

The “\( \in \)” symbol says the number \( n \) is an element of the set or interval given. The “\( \infty \)” symbol represents positive infinity and indicates the interval continues forever to the right. Note that the endpoints of an interval must occur in the same order as on the number line (smaller value on the left; larger value on the right).

A short summary of other possibilities is given here for any real number \( x \). Many variations are possible.

<table>
<thead>
<tr>
<th>Conditions ((a &lt; b))</th>
<th>Set Notation</th>
<th>Number Line</th>
<th>Interval Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x ) is greater than ( k )</td>
<td>( {x</td>
<td>x &gt; k} )</td>
<td>[Number line with an open circle at ( k ) and an arrow to the right]</td>
</tr>
<tr>
<td>( x ) is less than or equal to ( k )</td>
<td>( {x</td>
<td>x \leq k} )</td>
<td>[Number line with a closed circle at ( k ) and an arrow to the left]</td>
</tr>
<tr>
<td>( x ) is less than ( b ) and greater than ( a )</td>
<td>( {x</td>
<td>a &lt; x &lt; b} )</td>
<td>[Number line with an open circle at ( a ) and a closed circle at ( b ), an arrow to the right]</td>
</tr>
<tr>
<td>( x ) is less than ( b ) and greater than or equal to ( a )</td>
<td>( {x</td>
<td>a \leq x &lt; b} )</td>
<td>[Number line with a closed circle at ( a ) and an open circle at ( b ), an arrow to the right]</td>
</tr>
<tr>
<td>( x ) is less than ( a ) or ( x ) is greater than ( b )</td>
<td>( {x</td>
<td>x &lt; a \ \text{or} \ x &gt; b} )</td>
<td>[Number line with an open circle at ( a ) and a closed circle at ( b ), arrows to the left and right]</td>
</tr>
</tbody>
</table>

For the graph of \( y = |x| \) (Figure 1.45), a vertical line will intersect the graph (or its infinite extension) for all values of \( x \), and the domain is \( x \in (-\infty, \infty) \). Using vertical lines in this way also affirms the domain of \( y = x - 1 \) (Section 1.1, Figure 1.5) is \( x \in (-\infty, \infty) \) while the domain of the relation \( x = |y| \) (Section 1.1, Figure 1.6) is \( x \in [0, \infty) \).

Range and Horizontal Boundary Lines

The range of a relation can be found using a horizontal “boundary line,” since it will associate a value on the y-axis with a point on the graph (if it exists). Simply visualize a horizontal line and move the line up or down until you determine the graph will always intersect the line, or will no longer intersect the line. This will give you the boundaries of the range. Mentally applying this idea to the graph of \( y = \sqrt{x} \) (Figure 1.46) shows the range is \( y \in [0, \infty) \). Although shaped very differently, a horizontal boundary line shows the range of \( y = |x| \) (Figure 1.45) is also \( y \in [0, \infty) \).

EXAMPLE 6  ➤ Determining the Domain and Range of a Function

Use a table of values to graph the functions defined by

- \( y = x^2 \)
- \( y = \sqrt{x} \)

Then use boundary lines to determine the domain and range of each.
a. For \( y = x^2 \), it seems convenient to use inputs from \( x = -3 \) to \( x = 3 \), producing the following table and graph. Note the result is a basic parabola that “opens upward” (both ends point in the positive \( y \) direction), with a vertex at \((0, 0)\). Figure 1.47 shows a vertical line will intersect the graph or its extension anywhere it is placed. The domain is \( x \in (-\infty, \infty) \). Figure 1.48 shows a horizontal line will intersect the graph only for values of \( y \) that are greater than or equal to 0. The range is \( y \in [0, \infty) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

b. For \( y = \sqrt[3]{x} \), we select points that are perfect cubes where possible, then a few others to round out the graph. The resulting table and graph are shown. Notice there is a “pivot point” at \((0, 0)\) called a point of inflection, and the ends of the graph point in opposite directions. Figure 1.49 shows a vertical line will intersect the graph or its extension anywhere it is placed. Figure 1.50 shows a horizontal line will likewise always intersect the graph. The domain is \( x \in (-\infty, \infty) \), and the range is \( y \in (-\infty, \infty) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \sqrt[3]{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-8</td>
<td>(-2)</td>
</tr>
<tr>
<td>-4</td>
<td>(-1.6)</td>
</tr>
<tr>
<td>-1</td>
<td>(-1)</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>(\approx 1.6)</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

Implied Domains

When stated in equation form, the domain of a function is implicitly given by the expression used to define it, since the expression will dictate what input values are allowed. The implied domain is the set of all real numbers for which the function represents a real number. If the function involves a rational expression, the domain will exclude any input that causes a denominator of zero, since division by zero is undefined. If the function involves a square root expression, the domain will exclude inputs that create a negative radicand, since \( \sqrt{A} \) represents a real number only when \( A \geq 0 \).
**EXAMPLE 7**

Determining Implied Domains

State the domain of each function using interval notation.

a. \( y = \frac{3}{x + 2} \)  
   b. \( y = \sqrt{2x + 3} \)  
   c. \( y = \frac{x - 1}{x^2 - 9} \)  
   d. \( y = x^2 - 5x + 7 \)

**Solution**

- **a.** By inspection, we note an \( x \)-value of \(-2\) results in a zero denominator and must be excluded. The domain is \( x \in (-\infty, -2) \cup (-2, \infty) \).
- **b.** Since the radicand must be nonnegative, we solve the inequality \( 2x + 3 \geq 0 \), giving \( x \geq -\frac{3}{2} \). The domain is \( x \in \left[ -\frac{3}{2}, \infty \right) \).
- **c.** To prevent division by zero, inputs of \(-3\) and \(3\) must be excluded (set \( x^2 - 9 = 0 \) and solve by factoring). The domain is \( x \in (-\infty, -3) \cup (-3, 3) \cup (3, \infty) \). Note that \( x = 1 \) is in the domain since \( \frac{0}{8} = 0 \) is defined. See Figure 1.51, where \( Y_1 = \frac{X - 1}{X^2 - 9} \).
- **d.** Since squaring a number and multiplying a number by a constant are defined for all real numbers, the domain is \( x \in (-\infty, \infty) \).

**Now try Exercises 63 through 80**

**EXAMPLE 8**

Determining Implied Domains

Determine the domain of each function:

a. \( y = \sqrt{\frac{7}{x + 3}} \)  
   b. \( y = \frac{2x}{\sqrt{4x + 5}} \)

**Solution**

- **a.** For \( y = \sqrt{\frac{7}{x + 3}} \), we must have \( \frac{7}{x + 3} \geq 0 \) (for the radicand) and \( x + 3 \neq 0 \) (for the denominator). Since the numerator is always positive, we need \( x + 3 > 0 \), which gives \( x > -3 \). The domain is \( x \in (-3, \infty) \).

- **b.** For \( y = \frac{2x}{\sqrt{4x + 5}} \), we must have \( 4x + 5 \geq 0 \) and \( \sqrt{4x + 5} \neq 0 \). This shows we need \( 4x + 5 > 0 \), so \( x > -\frac{5}{4} \). The domain is \( x \in \left( -\frac{5}{4}, \infty \right) \).

**Now try Exercises 81 through 96**

**C. Function Notation**

In our study of functions, you've likely noticed that the relationship between input and output values is an important one. To highlight this fact, think of a function as a simple machine, which can process inputs using a stated sequence of operations, then deliver a single output. The inputs are \( x \)-values, a program we'll name \( f \) performs the operations on \( x \), and \( y \) is the resulting output (see Figure 1.52). Once again we see that "the value of \( y \) depends on the value of \( x \)," or simply "\( y \) is a function of \( x \)." Notationally, we write "\( y \) is a function of \( x \)" as \( y = f(x) \) using function notation. You are already familiar with letting a variable represent a number. Here we do something quite different, as the letter \( f \) is used to represent a sequence of operations to be performed on \( x \). Consider the function \( y = \frac{x}{2} + 1 \), which we'll now write as \( f(x) = \frac{x}{2} + 1 \) [since \( y = f(x) \)].
In words the function says, "divide inputs by 2, then add 1." To evaluate the function at \( x = 4 \) (Figure 1.53) we have:

\[
\begin{align*}
f(x) &= \frac{x}{2} + 1 \\
f(4) &= \frac{4}{2} + 1 \\
     &= 2 + 1 \\
     &= 3
\end{align*}
\]

Function notation enables us to summarize the three most important aspects of a function using a single expression, as shown in Figure 1.54.

Instead of saying, "... when \( x = 4 \), the value of the function is 3," we simply say "\( f \) of 4 is 3," or write \( f(4) = 3 \). Note that the ordered pair \((4, 3)\) is equivalent to \((4, f(4))\).

**CAUTION** Although \( f(x) \) is the favored notation for a "function of \( x \)," other letters can also be used. For example, \( g(x) \) and \( h(x) \) also denote functions of \( x \), where \( g \) and \( h \) represent different sequences of operations on the \( x \)-inputs. It is also important to remember that these represent function values and not the product of two variables: \( f(x) \neq f \cdot (x) \).

**EXAMPLE 9**

Evaluating a Function

Given \( f(x) = -2x^2 + 4x \), find

\[
\begin{align*}
a. \ f(-2) & \quad b. \ f\left(\frac{7}{2}\right) \\
c. \ f(2a) & \quad d. \ f(a + 1)
\end{align*}
\]

**Solution**

\[
\begin{align*}
a. \ f(-2) &= -2(-2)^2 + 4(-2) \\
       &= -8 + (-8) \\
       &= -16
\end{align*}
\]

\[
\begin{align*}
b. \ f\left(\frac{7}{2}\right) &= -2\left(\frac{7}{2}\right)^2 + 4\left(\frac{7}{2}\right) \\
       &= -2\left(\frac{49}{4}\right) + 4\left(\frac{7}{2}\right) \\
       &= -\frac{49}{2} + 14 \\
       &= -\frac{21}{2} \text{ or } -10.5
\end{align*}
\]

\[
\begin{align*}
c. \ f(2a) &= -2(2a)^2 + 4(2a) \\
        &= -2(4a^2) + 8a \\
        &= -8a^2 + 8a
\end{align*}
\]

\[
\begin{align*}
d. \ f(a + 1) &= -2(a + 1)^2 + 4(a + 1) \\
             &= -2(a^2 + 2a + 1) + 4a + 4 \\
             &= -2a^2 - 4a - 2 + 4a + 4 \\
             &= -2a^2 + 2
\end{align*}
\]

Now try Exercises 87 through 102.
A graphing calculator can evaluate the function \( Y_1 = -2x^2 + 4x \) using the TABLE feature, the \( \mathbb{C} \) feature, or function notation (on the home screen). The first two have been illustrated previously. To use function notation, we access the function names using the \( \mathbb{Y} \) key and right arrow to select \( Y-VARS \) (Figure 1.55). The \textbf{1:Function} option is the default, so pressing \( \mathbb{C} \) will enable us to make our choice (Figure 1.56). In this case, we selected \textbf{1:Y1}, which the calculator then places on the home screen, enabling us to enclose the desired input value in parentheses (function notation). Pressing ENTER completes the evaluation (Figure 1.57), which verifies the result from Example 9(b).

![Figure 1.55](image)

**D. Reading and Interpreting Information Given Graphically**

Graphs are an important part of studying functions, and learning to read and interpret them correctly is a high priority. A graph highlights and emphasizes the all-important input/output relationship that defines a function. In this study, we hope to firmly establish that the following statements are synonymous:

1. \( f(-2) = 5 \)
2. \( (-2, f(-2)) = (-2, 5) \)
3. \( (-2, 5) \) is on the graph of \( f \), and
4. when \( x = -2, f(x) = 5 \)

**EXAMPLE 10A ➤ Reading a Graph**

For the functions \( f \) and \( g \) whose graphs are shown in Figures 1.58 and 1.59

- **a.** State the domain of the function.
- **b.** Evaluate the function at \( x = 2 \).
- **c.** Determine the value(s) of \( x \) for which \( y = 3 \).
- **d.** State the range of the function.

![Figure 1.58 and 1.59](image)

**Solution ➤** For \( f \),

- **a.** The graph is a continuous line segment with endpoints at \((-4, -3)\) and \((5, 3)\), so we state the domain in interval notation. Using a vertical boundary line we note the smallest input is \(-4\) and the largest is \(5\). The domain is \( x \in [-4, 5] \).
b. The graph shows an input of \( x = 2 \) corresponds to \( y = 1 \): \( f(2) = 1 \) since \( (2, 1) \) is a point on the graph.

c. For \( f(x) = 3 \) (or \( y = 3 \)) the input value must be \( x = 5 \) since \( (5, 3) \) is the point on the graph.

d. Using a horizontal boundary line, the smallest output value is \(-3\) and the largest is \(3\). The range is \( y \in [-3, 3] \).

For \( g \):

a. Since \( g \) is given as a set of plotted points, we state the domain as the set of first coordinates: \( D: \{-4, -2, 0, 2, 4\} \).

b. An input of \( x = 2 \) corresponds to \( y = 2 \): \( g(2) = 2 \) since \( (2, 2) \) is on the graph.

c. For \( g(x) = 3 \) (or \( y = 3 \)) the input value must be \( x = 4 \), since \( (4, 3) \) is a point on the graph.

d. The range is the set of all second coordinates: \( R: \{-1, 0, 1, 2, 3\} \).

---

**EXAMPLE 10B**

**Reading a Graph**

Use the graph of \( f(x) \) given to answer the following questions:

a. What is the value of \( f(-2) \)?

b. What value(s) of \( x \) satisfy \( f(x) = 1 \)?

**Solution**

a. The notation \( f(-2) \) says to find the value of the function \( f \) when \( x = -2 \). Expressed graphically, we go to \( x = -2 \) and locate the corresponding point on the graph (blue arrows). Here we find that \( f(-2) = 4 \).

b. For \( f(x) = 1 \), we're looking for \( x \)-inputs that result in an output of \( y = 1 \) [since \( y = f(x) \)]. From the graph, we note there are two points with a \( y \)-coordinate of \( 1 \), namely, \((-3, 1)\) and \((0, 1)\). This shows \( f(-3) = 1, f(0) = 1 \), and the required \( x \)-values are \( x = -3 \) and \( x = 0 \).

---

In many applications involving functions, the domain and range can be determined by the context or situation given.

---

**EXAMPLE 11**

**Determining the Domain and Range from the Context**

Paul's 2009 Voyager has a 20-gal tank and gets 18 mpg. The number of miles he can drive (his range) depends on how much gas is in the tank. As a function we have \( M(g) = 18g \), where \( M(g) \) represents the total distance in miles and \( g \) represents the gallons of gas in the tank (see graph). Find the domain and range.

**Solution**

Begin evaluating at \( x = 0 \), since the tank cannot hold less than zero gallons. With an empty tank, the (minimum) range is \( M(0) = 18(0) \) or 0 miles. On a full tank, the maximum range is \( M(20) = 18(20) \) or 360 miles. As shown in the graph, the domain is \( g \in [0, 20] \) and the corresponding range is \( M(g) \in [0, 360] \).

---

D. You've just seen how we can read and interpret information given graphically.
1.3 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. If a relation is given in ordered pair form, we state the domain by listing all of the ______ coordinates in a set.

2. A relation is a function if each element of the ______ is paired with ______ ______ element of the range.

3. The set of output values for a function is called the ______ of the function.

4. Write using function notation: The function $f$ evaluated at 3 is negative 5: ______

5. Discuss/Explain why the relation $y = x^2$ is a function, while the relation $x = y^2$ is not. Justify your response using graphs, ordered pairs, and so on.

6. Discuss/Explain the process of finding the domain and range of a function given its graph, using vertical and horizontal boundary lines. Include a few illustrative examples.

DEVELOPING YOUR SKILLS

Determine whether the mappings shown represent functions or nonfunctions. If a nonfunction, explain how the definition of a function is violated.

7. Woman

<table>
<thead>
<tr>
<th>Indira Gandhi</th>
<th>Margaret Thatcher</th>
<th>Maria Montessori</th>
<th>Susan B. Anthony</th>
</tr>
</thead>
<tbody>
<tr>
<td>Britain</td>
<td>Italy</td>
<td>India</td>
<td></td>
</tr>
</tbody>
</table>

Country

<table>
<thead>
<tr>
<th>Canada</th>
<th>Japan</th>
<th>Brazil</th>
<th>Tahiti</th>
<th>Ecuador</th>
</tr>
</thead>
<tbody>
<tr>
<td>Japanese</td>
<td>Spanish</td>
<td>French</td>
<td>Portuguese</td>
<td>English</td>
</tr>
</tbody>
</table>

8. Book

<table>
<thead>
<tr>
<th>Hawaii Roots</th>
<th>Shogun</th>
<th>20,000 Leagues Under the Sea</th>
<th>Where the Red Fern Grows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rawls</td>
<td>Verne</td>
<td>Haley</td>
<td>Clavell</td>
</tr>
</tbody>
</table>

Author

| Michener |

9. Basketball star

<table>
<thead>
<tr>
<th>Air Jordan</th>
<th>The Mailman</th>
<th>The Doctor</th>
<th>The Iceman</th>
<th>The Shaq</th>
</tr>
</thead>
<tbody>
<tr>
<td>7'1&quot;</td>
<td>6'6&quot;</td>
<td>6'7&quot;</td>
<td>6'9&quot;</td>
<td>7'2&quot;</td>
</tr>
</tbody>
</table>

Reported height

Determine whether the relations indicated represent functions or nonfunctions. If the relation is a nonfunction, explain how the definition of a function is violated.

10. Country

<table>
<thead>
<tr>
<th>Labor</th>
<th>Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>Canada</td>
<td>Japanese</td>
</tr>
<tr>
<td>Japan</td>
<td>Spanish</td>
</tr>
<tr>
<td>Brazil</td>
<td>French</td>
</tr>
<tr>
<td>Tahiti</td>
<td>Portuguese</td>
</tr>
<tr>
<td>Ecuador</td>
<td>English</td>
</tr>
</tbody>
</table>

11. $(-3, 0), (1, 4), (2, -5), (4, 2), (-5, 6), (3, 6), (0, -1), (4, -5), and (6, 1)$

12. $(-7, -5), (-5, 3), (4, 0), (-3, -5), (1, -6), (0, 9), (2, -8), (3, -2), and (-5, 7)$

13. $(9, -10), (-7, 6), (6, -10), (4, -1), (2, -2), (1, 8), (0, -2), (-2, -7), and (-6, 4)$

14. $(1, -81), (-2, 64), (-3, 49), (5, -36), (-8, 25), (13, -16), (-21, 9), (34, -4), and (-55, 1)$

15. 

16. 

Determine whether the relations indicated represent functions or nonfunctions. If the relation is a nonfunction, explain how the definition of a function is violated.
Determine whether or not the relations given represent a function. If not, explain how the definition of a function is violated.

Graph each relation using a table, then use the vertical line test to determine if the relation is a function.

31. \( y = x \)  
32. \( y = \sqrt{x} \)
33. \( y = (x + 2)^2 \)  
34. \( x = |y - 2| \)

Use an inequality to write a mathematical model for each statement, then write the relation in interval notation.

35. To qualify for a secretarial position, a person must type at least 45 words per minute.
36. The balance in a checking account must remain above $1000 or a fee is charged.
37. To bake properly, a turkey must be kept between the temperatures of 250° and 450°.
38. To fly effectively, the airliner must cruise at or between altitudes of 30,000 and 35,000 ft.

Graph each inequality on a number line, then write the relation in interval notation.

39. \( p < 3 \)  
40. \( x > -2 \)
41. \( m \leq 5 \)  
42. \( n \geq -4 \)
43. \( x \neq 1 \)  
44. \( x \neq -3 \)
45. \( 5 > x > 2 \)  
46. \( -3 < p \leq 4 \)

Write the domain illustrated on each graph in set notation and interval notation.

47. \([-3, -2, 0, 1, 2, 3]\)  
48. \([-3, -2, 0, 1, 2, 3]\)
49. \([-3, -2, -1, 0, 1, 2, 3]\)
50. \([-3, -2, -1, 0, 1, 2, 3, 4]\)

Determine whether or not the relations indicated represent functions, then determine the domain and range of each.

51. 
52. 
53. $m = n^2 - 3n - 10$  
54. $s = r^2 - 3t - 10$
55. $y = 2|x| + 1$  
56. $y = |x - 2| + 3$
57. $y_1 = \frac{x}{x^2 - 2x - 10}$  
58. $y_2 = \frac{x - 4}{x^2 + 2x - 15}$
59. $y = \frac{\sqrt{x - 2}}{2x - 5}$  
60. $y = \frac{\sqrt{x + 1}}{3x + 2}$
61. $h(x) = \frac{-2}{\sqrt{x + 4}}$  
62. $f(x) = \frac{5}{\sqrt{x - 2}}$
63. $g(x) = \sqrt{3a + 5}$  
64. $p(x) = \sqrt{5a - 2}$
65. $v(x) = \frac{x + 2}{x^2 - 25}$  
66. $w(x) = \frac{x - 4}{x^2 - 49}$
67. $u = \frac{\sqrt{x - 5}}{\sqrt{x^2 - 18}}$  
68. $p = \frac{q + 7}{q^2 - 12}$
69. $y = \frac{17}{25x} + 123$  
70. $y = \frac{11}{19x} - 89$
71. $m = n^2 - 3n - 10$  
72. $s = r^2 - 3t - 10$
73. $y = 2|x| + 1$  
74. $y = |x - 2| + 3$
75. $y_1 = \frac{x}{x^2 - 2x - 10}$  
76. $y_2 = \frac{x - 4}{x^2 + 2x - 15}$
77. $y = \frac{\sqrt{x - 2}}{2x - 5}$  
78. $y = \frac{\sqrt{x + 1}}{3x + 2}$
79. $h(x) = \frac{-2}{\sqrt{x + 4}}$  
80. $f(x) = \frac{5}{\sqrt{x - 2}}$
81. $g(x) = \sqrt{3a + 5}$  
82. $p(x) = \sqrt{5a - 2}$
83. $v(x) = \frac{x + 2}{x^2 - 25}$  
84. $w(x) = \frac{x - 4}{x^2 - 49}$
85. $u = \frac{\sqrt{x - 5}}{\sqrt{x^2 - 18}}$  
86. $p = \frac{q + 7}{q^2 - 12}$
87. $y = \frac{17}{25x} + 123$  
88. $y = \frac{11}{19x} - 89$
89. $m = n^2 - 3n - 10$  
90. $s = r^2 - 3t - 10$
91. $y = 2|x| + 1$  
92. $y = |x - 2| + 3$
93. $y_1 = \frac{x}{x^2}$  
94. $y_2 = \frac{2x - 1}{\sqrt{3x - 7}}$
95. $h(x) = \frac{3}{x}$  
96. $h(x) = \frac{2}{x^2}$
97. $h(x) = \frac{5|x|}{x}$  
98. $h(x) = \frac{4|x|}{x}$
99. $g(x) = \sqrt{3a + 5}$  
100. $p(x) = \sqrt{5a - 2}$
101. $p(x) = \frac{x + 2}{x^2}$  
102. $p(x) = \frac{q + 7}{q^2 - 12}$

For Exercises 87 through 102, determine the value of \( f(-6), f(\frac{2}{3}), f(2c), \) and \( f(c + 1) \), then simplify. Verify results using a graphing calculator where possible.

87. \( f(x) = \frac{1}{2}x + 3 \)  
88. \( f(x) = \frac{2}{3}x - 5 \)
89. \( f(x) = 3x^2 - 4x \)  
90. \( f(x) = 2x^2 + 3x \)

Determine the value of \( h(3), h(-\frac{3}{4}), h(3a), \) and \( h(a - 2) \), then simplify.

91. \( h(x) = \frac{3}{x} \)  
92. \( h(x) = \frac{2}{x^2} \)
93. \( h(x) = \frac{5|x|}{x} \)  
94. \( h(x) = \frac{4|x|}{x} \)

Determine the value of \( g(4), g(\frac{2}{3}), g(2c), \) and \( g(c + 3) \), then simplify.

95. \( g(r) = 2\pi r \)  
96. \( g(r) = 2\pi rh \)
97. \( g(r) = \pi r^2 \)  
98. \( g(r) = \pi r^2h \)

Determine the value of \( p(5), p(\frac{2}{3}), p(3a), \) and \( p(a - 1) \), then simplify.

99. \( p(x) = \sqrt{2x + 3} \)  
100. \( p(x) = \sqrt{4x - 1} \)
101. \( p(x) = \frac{3x^2 - 5}{x^2} \)  
102. \( p(x) = \frac{2x^2 + 3}{x^2} \)

Use the graph of each function given to (a) state the domain, (b) state the range, (c) evaluate \( f(2) \), and (d) find the value(s) \( x \) for which \( f(x) = k \) (k a constant). Assume all results are integer-valued.

103. \( k = 4 \)  
104. \( k = 3 \)
109. Ideal weight for males: \( W(H) = \frac{9}{2}H - 151 \)

The ideal weight for an adult male can be modeled by the function shown, where \( W \) is his weight in pounds and \( H \) is his height in inches. (a) Find the ideal weight for a male who is 75 in. tall. (b) If I am 72 in. tall and weigh 210 lb, how much weight should I lose?

110. Celsius to Fahrenheit conversions: \( C = \frac{5}{9}(F - 32) \)

The relationship between Fahrenheit degrees and degrees Celsius is modeled by the function shown. (a) What is the Celsius temperature if \( ^\circ F = 41 \) ? (b) Use the formula to solve for \( F \) in terms of \( C \), then substitute the result from part (a). What do you notice?

111. Pick’s theorem: \( A = \frac{1}{2}B + I - 1 \)

Pick’s theorem is an interesting yet little known formula for computing the area of a polygon drawn in the Cartesian coordinate system. The formula can be applied as long as the vertices of the polygon are lattice points (both \( x \) and \( y \) are integers). If \( B \) represents the number of lattice points lying directly on the boundary of the polygon (including the vertices), and \( I \) represents the number of points in the interior, the area of the polygon is given by the formula shown. Use some graph paper to carefully draw a triangle with vertices at \((-3, 1), (3, 9), \) and \((7, 6)\), then use Pick’s theorem to compute the triangle’s area.

112. Gas mileage: John’s old ’87 LeBaron has a 15-gal gas tank and gets 23 mpg. The number of miles he can drive is a function of how much gas is in the tank. (a) Write this relationship in equation form and (b) determine the domain and range of the function in this context.

113. Gas mileage: Jackie has a gas-powered model boat with a 5-oz gas tank. The boat will run for 2.5 min on each ounce. The number of minutes she can operate the boat is a function of how much gas is in the tank. (a) Write this relationship in equation form and (b) determine the domain and range of the function in this context.

114. Volume of a cube: The volume of a cube depends on the length of the sides. In other words, volume is a function of the sides: \( V(s) = s^3 \). (a) In practical terms, what is the domain of this function? (b) Evaluate \( V(6.25) \) and (c) evaluate the function for \( s = 2^\frac{1}{3} \).

115. Volume of a cylinder: For a fixed radius of 10 cm, the volume of a cylinder depends on its height. In other words, volume is a function of height:

\[ V(h) = 100\pi h. \] (a) In practical terms, what is the domain of this function? (b) Evaluate \( V(7.5) \) and (c) evaluate the function for \( h = \frac{8}{\pi} \).

116. Rental charges: Temporary Transportation Inc. rents cars (local rentals only) for a flat fee of \$19.50 and an hourly charge of \$12.50. This means that cost is a function of the hours the car is rented plus the flat fee. (a) Write this relationship in equation form; (b) find the cost if the car is rented for 3.5 hr; (c) determine how long the car was rented if the bill came to \$119.75; and (d) determine the domain and range of the function in this context, if your budget limits you to paying a maximum of \$150 for the rental.

117. Cost of a service call: Paul’s Plumbing charges a flat fee of \$50 per service call plus an hourly rate of \$42.50. This means that cost is a function of the hours the job takes to complete plus the flat fee. (a) Write this relationship in equation form; (b) find the cost of a service call that takes \( 2\frac{1}{2} \) hr; (c) find the number of hours the job took if the charge came to \$262.50; and (d) determine the
domain and range of the function in this context, if your insurance company has agreed to pay for all charges over $500 for the service call.

118. Predicting tides: The graph shown approximates the height of the tides at Fair Haven, New Brunswick, for a 12-hr period. (a) Is this the graph of a function? Why? (b) Approximately what time did high tide occur? (c) How high is the tide at 6 P.M.? (d) What time(s) will the tide be 2.5 m?

119. Predicting tides: The graph shown approximates the height of the tides at Apia, Western Samoa, for a 12-hr period. (a) Is this the graph of a function? Why? (b) Approximately what time did low tide occur? (c) How high is the tide at 2 A.M.? (d) What time(s) will the tide be 0.7 m?

EXTENDING THE CONCEPT

120. A father challenges his son to a 400-m race, depicted in the graph shown here.

```
Distance in meters
0  100  200  300  400
```

```
Time in seconds
10 20 30 40 50 60 70 80
```

Father: --- Son: ---

a. Who won and what was the approximate winning time?
b. Approximately how many meters behind was the second place finisher?
c. Estimate the number of seconds the father was in the lead in this race.
d. How many times during the race were the father and son tied?

121. Sketch the graph of \( f(x) = x \), then discuss how you could use this graph to obtain the graph of \( F(x) = |x| \) without computing additional points.

What would the graph of \( g(x) = \frac{|x|}{x} \) look like?

122. Sketch the graph of \( f(x) = x^2 - 4 \), then discuss how you could use this graph to obtain the graph of \( F(x) = |x^2 - 4| \) without computing additional points.

Determine what the graph of \( g(x) = \frac{|x^2 - 4|}{x^2 - 4} \) would look like.

123. If the equation of a function is given, the domain is implicitly defined by input values that generate real-valued outputs. But unless the graph is given or can be easily sketched, we must attempt to find the range analytically by solving for \( x \) in terms of \( y \). We should note that sometimes this is an easy task, while at other times it is virtually impossible and we must rely on other methods. For the following functions, determine the implicit domain and find the range by solving for \( x \) in terms of \( y \).

a. \( y = \frac{x - 3}{x + 2} \)

b. \( y = x^2 - 3 \)

MAINTAINING YOUR SKILLS

124. (1.1) Find the equation of a circle whose center is \((4, -1)\) with a radius of 5. Then graph the circle.

125. (Appendix A.6) Compute the sum and product indicated:

a. \( \sqrt{24} + 6\sqrt{54} - \sqrt{6} \)
b. \((2 + \sqrt{3})(2 - \sqrt{3})\)

126. (Appendix A.4) Solve the equation by factoring, then check the root(s) using substitution:

\(3x^2 - 4x = 7\).

127. (Appendix A.4) Factor the following polynomials completely:

a. \(x^3 - 3x^2 - 25x + 75\)
b. \(2x^2 - 13x - 24\)
c. \(8x^3 - 125\)
1. Sketch the graph of the line $4x - 3y = 12$. Plot and label at least three points.

2. Find the slope of the line passing through the given points: $(-3, 8)$ and $(4, -10)$.

3. In 2009, Data.com lost $2$ million. In 2010, they lost $0.5$ million. Will the slope of the line through these points be positive or negative? Why? Calculate the slope. Were you correct? Write the slope as a unit rate and explain what it means in this context.

4. To earn some spending money, Sahara takes a job in a ski shop working primarily with her specialty—snowboards. She is paid a monthly salary of $950 plus a commission of $7.50 per snowboard she sells. (a) Write a function that models her monthly earnings $E$. (b) Use a graphing calculator to determine her income if she sells 20, 30, or 40 snowboards in one month. (c) Use the results of parts a and b to set an appropriate viewing window and graph the line. (d) Use the TRACE feature to determine the number of snowboards that must be sold for Sahara's monthly income to top $1300.

5. Write the equation for line $L_2$ shown. Is this the graph of a function? Discuss why or why not.

6. Write the equation for line $L_3$ shown. Is this the graph of a function? Discuss why or why not.

7. For the graph of function $h(x)$ shown, (a) determine the value of $h(2)$; (b) state the domain; (c) determine the value(s) of $x$ for which $h(x) = -3$; and (d) state the range.

8. Judging from the appearance of the graph alone, compare the rate of change (slope) from $x = 1$ to $x = 2$ to the rate of change from $x = 4$ to $x = 5$. Which rate of change is larger? How is that demonstrated graphically?

9. Compute the slope of the line shown, and explain what it means as a rate of change in this context. Then use the slope to predict the fox population when the pheasant population is $13,000$.

10. State the domain and range for each function below.

**REINFORCING BASIC CONCEPTS**

Finding the Domain and Range of a Relation from Its Graph

The concepts of domain and range are an important and fundamental part of working with relations and functions. In this chapter, we learned to determine the domain of any relation from its graph using a "vertical boundary line," and the range by using a "horizontal boundary line." These approaches to finding the domain and range can be combined into a single step by envisioning a rectangle drawn around or about the graph. If the entire graph can be "bounded" within the rectangle, the domain and range can be based on the rectangle's related length and width. If it's impossible to bound the graph in a particular direction, the related $x$- or $y$-values continue infinitely. Consider the graph in Figure 1.60. This is the graph of an ellipse (Section 8.2), and a rectangle that bounds the graph in all directions is shown in Figure 1.61.
The rectangle extends from $x = -3$ to $x = 9$ in the horizontal direction, and from $y = 1$ to $y = 7$ in the vertical direction. The domain of this relation is $x \in [-3, 9]$ and the range is $y \in [1, 7]$.

The graph in Figure 1.62 is a parabola, and no matter how large we draw the rectangle, an infinite extension of the graph will extend beyond its boundaries in the left and right directions, and in the upward direction (Figure 1.63).

The domain of this relation is $x \in (-\infty, \infty)$ and the range is $y \in [-6, \infty)$.

Finally, the graph in Figure 1.64 is the graph of a square root function, and a rectangle can be drawn that bounds the graph below and to the left, but not above or to the right (Figure 1.65).

The domain of this relation is $x \in [-7, \infty)$ and the range is $y \in [-5, \infty)$.

Use this approach to find the domain and range of the following relations and functions.

Exercise 1:

Exercise 2:

Exercise 3:

Exercise 4:
1.4 Linear Functions, Special Forms, and More on Rates of Change

LEARNING OBJECTIVES
In Section 1.4 you will see how we can:

- **A.** Write a linear equation in slope-intercept form and function form
- **B.** Use slope-intercept form to graph linear equations
- **C.** Write a linear equation in point-slope form
- **D.** Apply the slope-intercept form and point-slope form in context

The concept of slope is an important part of mathematics, because it gives us a way to measure and compare change. The value of an automobile changes with time, the circumference of a circle increases as the radius increases, and the tension in a spring grows the more it is stretched. The real world is filled with examples of how one change affects another, and slope helps us understand how these changes are related.

A. Linear Equations, Slope-Intercept Form and Function Form

In Section 1.2, we learned that a linear equation is one that can be written in the form \( ax + by = c \). Solving for \( y \) in a linear equation offers distinct advantages to understanding linear graphs and their applications.

**EXAMPLE 1**

**Solving for \( y \) in a Linear Equation**

Solve \( 2y - 6x = 4 \) for \( y \), then evaluate at \( x = 4, x = 0, \) and \( x = -\frac{1}{3} \).

**Solution**

\[
\begin{align*}
2y - 6x &= 4 & \text{given equation} \\
2y &= 6x + 4 & \text{add } 6x \\
y &= 3x + 2 & \text{divide by } 2
\end{align*}
\]

Since the coefficients are integers, evaluate the function mentally. Inputs are multiplied by 3, then increased by 2, yielding the ordered pairs \((4, 14), (0, 2), \) and \((-\frac{1}{3}, 1)\).

Now try Exercises 7 through 12

This form of the equation (where \( y \) has been written in terms of \( x \)) enables us to quickly identify what operations are performed on \( x \) in order to obtain \( y \). Once again, for \( y = 3x + 2 \): multiply inputs by 3, then add 2.

**EXAMPLE 2**

**Solving for \( y \) in a Linear Equation**

Solve the linear equation \( 3y - 2x = 6 \) for \( y \), then identify the new coefficient of \( x \) and the constant term.

**Solution**

\[
\begin{align*}
3y - 2x &= 6 & \text{given equation} \\
3y &= 2x + 6 & \text{add } 2x \\
y &= \frac{2}{3}x + 2 & \text{divide by } 3
\end{align*}
\]

The coefficient of \( x \) is \( \frac{2}{3} \) and the constant term is 2.

Now try Exercises 13 through 18

When the coefficient of \( x \) is rational, it’s helpful to select inputs that are multiples of the denominator if the context or application requires us to evaluate the equation. This enables us to perform most operations mentally. For \( y = \frac{2}{3}x + 2 \), possible inputs might be \( x = -9, -6, 0, 3, 6 \), and so on. See Exercises 19 through 24.

In Section 1.2, linear equations were graphed using the intercept method. When the equation is written with \( y \) in terms of \( x \), we notice a powerful connection between the graph and its equation—one that highlights the primary characteristics of a linear graph.
EXAMPLE 3  Noting Relationships between an Equation and Its Graph

Find the intercepts of \(4x + 5y = -20\) and use them to graph the line. Then,

a. Use the intercepts to calculate the slope of the line, then identify the 
   \(y\)-intercept.

b. Write the equation with \(y\) in terms of \(x\) and compare the calculated slope and 
   \(y\)-intercept to the equation in this form. Comment on what you notice.

Solution  Substituting 0 for \(x\) in \(4x + 5y = -20\), we find the
   \(y\)-intercept is \((0, -4)\). Substituting 0 for \(y\) gives an 
   \(x\)-intercept of \((-5, 0)\). The graph is displayed here.

a. The \(y\)-intercept is \((0, -4)\) and by calculation or
   counting \(\frac{\Delta y}{\Delta x}\), the slope is \(m = \frac{-4}{5}\) [from the
   intercept \((-5, 0)\) we count down 4, giving 
   \(\Delta y = -4\), and right 5, giving \(\Delta x = 5\), to
   arrive at the intercept \((0, -4)\)].

b. Solving for \(y\):
   
   \[
   \begin{align*}
   4x + 5y &= -20 \quad \text{given equation} \\
   5y &= -4x - 20 \quad \text{subtract 4x} \\
   y &= \frac{-4}{5}x - 4 \quad \text{divide by 5}
   \end{align*}
   \]

   The slope value seems to be the coefficient of \(x\), while the \(y\)-intercept is the
   constant term.

Now try Exercises 25 through 30

After solving a linear equation for \(y\), an input of \(x = 0\) causes the "\(x\)-term" to become 
zero, so the \(y\)-intercept automatically involves the constant term. As Example 3 illustrates,
we can also identify the slope of the line—it is the coefficient of \(x\). In general, a linear 
equation of the form \(y = mx + b\) is said to be in slope-intercept form, since the slope 
of the line is \(m\) and the \(y\)-intercept is \((0, b)\).

Slope-Intercept Form

For a nonvertical line whose equation is \(y = mx + b\),
the slope of the line is \(m\) and the \(y\)-intercept is \((0, b)\).

Solving a linear equation for \(y\) in terms of \(x\) is sometimes called writing the equation in 
function form, as this form clearly highlights what operations are performed on 
the input value in order to obtain the output (see Example 1). In other words, this form 
plainly shows that "\(y\) depends on \(x\)," or "\(y\) is a function of \(x\)," and that the equations 
\(y = mx + b\) and \(f(x) = mx + b\) are equivalent.

Linear Functions

A linear function is one of the form 

\[
   f(x) = mx + b,
\]

where \(m\) and \(b\) are real numbers.

Note that if \(m = 0\), the result is a constant function \(f(x) = b\). If \(m = 1\) and \(b = 0\), the 
result is \(f(x) = x\), called the identity function.
EXAMPLE 4 ▶ Finding the Function Form of a Linear Equation

Write each equation in both slope-intercept form and function form. Then identify the slope and y-intercept of the line.

a. \(3x - 2y = 9\)  
   \[\begin{align*}
   -2y &= -3x + 9 \\
   y &= \frac{3x}{2} - \frac{9}{2}
   \end{align*}\]
   \[\begin{align*}
   f(x) &= \frac{3}{2}x - \frac{9}{2} \\
   m &= \frac{3}{2}, b = -\frac{9}{2}
   \end{align*}\]
   y-intercept \((0, -\frac{9}{2})\)

b. \(y + x = 5\)  
   \[\begin{align*}
   y &= -x + 5 \\
   f(x) &= -1x + 5 \\
   m &= -1, b = 5
   \end{align*}\]
   y-intercept \((0, 5)\)

c. \(2y = x\)  
   \[\begin{align*}
   y &= \frac{x}{2} \\
   f(x) &= \frac{1}{2}x \\
   m &= \frac{1}{2}, b = 0
   \end{align*}\]
   y-intercept \((0, 0)\)

A. You've just seen how we can write a linear equation in slope-intercept form and function form

Note that we can analytically develop the slope-intercept form of a line using the slope formula. Figure 1.66 shows the graph of a general line through the point \((x, y)\) with a y-intercept of \((0, b)\). Using these points in the slope formula, we have

\[
\frac{y_2 - y_1}{x_2 - x_1} = m \quad \text{slope formula}
\]

\[
\frac{y - b}{x - 0} = m \quad \text{substitute: } (0, b) \text{ for } (x_1, y_1), (x, y) \text{ for } (x_2, y_2)
\]

\[
\frac{y - b}{x} = m \quad \text{simplify}
\]

\[
y - b = mx \quad \text{multiply by } x
\]

\[
y = mx + b \quad \text{add } b \text{ to both sides}
\]

This approach confirms the relationship between the graphical characteristics of a line and its slope-intercept form. Specifically, for any linear equation written in the form \(y = mx + b\), the slope must be \(m\) and the y-intercept is \((0, b)\).

B. Slope-Intercept Form and the Graph of a Line

If the slope and y-intercept of a linear equation are known or can be found, we can construct its equation by substituting these values directly into the slope-intercept form \(y = mx + b\).

EXAMPLE 5 ▶ Finding the Equation of a Line from Its Graph

Find the slope-intercept equation of the line shown.

Using \((-3, -2)\) and \((-1, 2)\) in the slope formula, or by simply counting \(\Delta y \over \Delta x\), the slope is \(m = \frac{3}{2}\) or \(\frac{3}{2}\).

By inspection we see the y-intercept is \((0, 4)\). Substituting \(\frac{3}{2}\) for \(m\) and 4 for \(b\) in the slope-intercept form we obtain the equation

\[y = \frac{3}{2}x + 4\]
Actually, if the slope is known and we have any point \((x, y)\) on the line, we can still construct the equation since the given point must satisfy the equation of the line. In this case, we’re treating \(y = mx + b\) as a simple formula, solving for \(b\) after substituting known values for \(m, x,\) and \(y.\)

### Example 6

**Using \(y = mx + b\) as a Formula**

Find the slope-intercept equation of a line that has slope \(m = \frac{3}{4}\) and contains \((-5, 2)\). Verify results on a graphing calculator.

**Solution**

Use \(y = mx + b\) as a “formula,” with \(m = \frac{3}{4}, x = -5,\) and \(y = 2.\)

\[
\begin{align*}
y &= mx + b & \text{slope-intercept form} \\
2 &= \frac{3}{4}(-5) + b & \text{substitute } \frac{3}{4} \text{ for } m, -5 \text{ for } x, \text{ and } 2 \text{ for } y \\
2 &= -4 + b & \text{simplify} \\
6 &= b & \text{solve for } b
\end{align*}
\]

The equation of the line is \(y = \frac{3}{4}x + 6.\) After entering the equation on the home screen of a graphing calculator, we can evaluate \(x = -5\) on the screen, or use the **trace** feature. See the figures provided.

Now try Exercises 45 through 60

Writing a linear equation in slope-intercept form enables us to draw its graph with a minimum of effort, since we can easily locate the \(y\)-intercept and a second point using the rate of change \(\frac{\Delta y}{\Delta x}\). For instance, \(\frac{\Delta y}{\Delta x} = \frac{-2}{3}\) indicates that counting down 2 and right 3 from a known point will locate another point on this line.

### Example 7

**Graphing a Line Using Slope-Intercept Form and the Rate of Change**

Write \(3y - 5x = 9\) in slope-intercept form, then graph the line using the \(y\)-intercept and the rate of change (slope).

\[
\begin{align*}
3y - 5x &= 9 & \text{given equation} \\
3y &= 5x + 9 & \text{isolate } y \text{ term} \\
y &= \frac{5}{3}x + 3 & \text{divide by } 3
\end{align*}
\]

The slope is \(m = \frac{5}{3}\) and the \(y\)-intercept is \((0, 3).\)

Plot the \(y\)-intercept, then use \(\frac{\Delta y}{\Delta x} = \frac{5}{3}\) (up 5 and right 3—shown in blue) to find another point on the line (shown in red). Finish by drawing a line through these points.

Now try Exercises 51 through 62

For a discussion of what graphing method might be most efficient for a given linear equation, see Exercises 103 and 114.

**Parallel and Perpendicular Lines**

From Section 1.2 we know parallel lines have equal slopes: \(m_1 = m_2,\) and perpendicular lines have slopes with a product of \(-1: m_1 \cdot m_2 = -1\) or \(m_1 = -\frac{1}{m_2}.\) In some applications, we need to find the equation of a second line parallel or perpendicular to a given line, through a given point. Using the slope-intercept form makes this a simple four-step process.
Finding the Equation of a Line Parallel or Perpendicular to a Given Line

1. Identify the slope \( m_1 \) of the given line.
2. Find the slope \( m_2 \) of the new line using the parallel or perpendicular relationship.
3. Use \( m_2 \) with the point \((x, y)\) in the "formula" \( y = mx + b \) and solve for \( b \).
4. The desired equation will be \( y = m_2 x + b \).

EXAMPLE 8 ➤ Finding the Equation of a Parallel Line

Find the slope-intercept equation of a line that goes through \((-6, -1)\) and is parallel to \(2x + 3y = 6\).

Solution ➤ Begin by writing the equation in slope-intercept form to identify the slope.

\[
2x + 3y = 6 \quad \text{given line}
\]
\[
3y = -2x + 6 \quad \text{isolate } y\text{-term}
\]
\[
y = \frac{-2}{3}x + 2 \quad \text{result}
\]

The original line has slope \( m_1 = \frac{-2}{3} \) and this will also be the slope of any line parallel to it. Using \( m_2 = \frac{-2}{3} \) with \((x, y) \rightarrow (-6, -1)\) we have

\[
y = mx + b \quad \text{slope-intercept form}
\]
\[
-1 = \frac{-2}{3}(-6) + b \quad \text{substitute } \frac{-2}{3} \text{ for } m,
\]
\[
-6 = 4 + b \quad \text{for } x, \text{ and } -1 \text{ for } y
\]
\[
-10 = b \quad \text{solve for } b
\]

The equation of the new line is \( y = \frac{-2}{3}x - 5 \).

Graphing the lines from Example 8 as \( Y_1 \) and \( Y_2 \) on a graphing calculator, we note the lines do appear to be parallel (they actually must be since they have identical slopes). Using the \( \textbf{Zoom} \#8: \text{ZInteger} \) feature of the calculator, we can quickly verify that \( Y_2 \) indeed contains the point \((-6, -1)\).

For any nonlinear graph, a straight line drawn through two points on the graph is called a \textit{secant line}. The slope of a secant line, and lines parallel and perpendicular to this line, play fundamental roles in the further development of the rate-of-change concept.

EXAMPLE 9 ➤ Finding Equations for Parallel and Perpendicular Lines

A secant line is drawn using the points \((-4, 0)\) and \((2, -2)\) on the graph of the function shown. Find the equation of a line that is

a. parallel to the secant line through \((-1, -4)\).

b. perpendicular to the secant line through \((-1, -4)\).

Solution ➤ Either by using the slope formula or counting \( \frac{\Delta y}{\Delta x} \), we find the secant line has slope

\[
m = \frac{-2}{6} = \frac{-1}{3}.
\]
a. For the parallel line through \((-1, -4), m_2 = -\frac{1}{3}\),

\[ y = mx + b \quad \text{slope-intercept form} \]

\[ -4 = -\frac{1}{3}(-1) + b \quad \text{substitute } -\frac{1}{3} \text{ for } m, \]

\[-1 \text{ for } x, \text{ and } -4 \text{ for } y \]

\[ \frac{12}{3} = \frac{1}{3} + b \quad \text{simplify } (-4 = -\frac{12}{3}) \]

\[ \frac{13}{3} = b \quad \text{result} \]

The equation of the parallel line (in blue) is \(y = -\frac{1}{3}x - \frac{13}{3}\).

b. For the perpendicular line through \((-1, -4), m_2 = 3\),

\[ y = mx + b \quad \text{slope-intercept form} \]

\[ -4 = 3(-1) + b \quad \text{substitute } 3 \text{ for } m, \text{ -1 for } x, \text{ and } -4 \text{ for } y \]

\[ -4 = -3 + b \quad \text{simplify} \]

\[ -1 = b \quad \text{result} \]

The equation of the perpendicular line (in yellow) is \(y = 3x - 1\).

C. Linear Equations in Point-Slope Form

As an alternative to using \(y = mx + b\), we can find the equation of the line using the slope formula \(\frac{y_2 - y_1}{x_2 - x_1} = m\), and the fact that the slope of a line is constant. For a given slope \(m\), we can let \((x_1, y_1)\) represent a given point on the line and \((x, y)\) represent any other point on the line, and the formula becomes \(\frac{y - y_1}{x - x_1} = m\). Isolating the “\(y\)” terms on one side gives a new form for the equation of a line, called the point-slope form:

\[ \frac{y - y_1}{x - x_1} = m \quad \text{slope formula} \]

\[ (y - y_1) = m(x - x_1) \quad \text{multiply both sides by } (x - x_1) \]

\[ y - y_1 = m(x - x_1) \quad \text{simplify } \rightarrow \text{ point-slope form} \]

The Point-Slope Form of a Linear Equation

For a nonvertical line whose equation is \(y - y_1 = m(x - x_1)\),

the slope of the line is \(m\) and \((x_1, y_1)\) is a point on the line.

While using \(y = mx + b\) (as in Example 6) may appear to be easier, both the slope-intercept form and point-slope form have their own advantages and it will help to be familiar with both.
EXAMPLE 10  Using \( y - y_1 = m(x - x_1) \) as a Formula

Find the equation of the line in point-slope form, if \( m = \frac{2}{3} \) and \((-3, -3)\) is on the line. Then graph the line.

**Solution**

\[
y - y_1 = m(x - x_1)
\]

point-slope form

\[
y - (-3) = \frac{2}{3}[x - (-3)]
\]

substitute \( \frac{2}{3} \) for \( m \); \((-3, -3)\)

for \((x_1, y_1)\)

\[
y + 3 = \frac{2}{3}(x + 3)
\]

simplify, point-slope form

To graph the line, plot \((-3, -3)\) and use \( \frac{\Delta y}{\Delta x} = \frac{2}{3} \) to find additional points on the line.

C. You’ve just seen how we can write a linear equation in point-slope form

Now try Exercises 83 through 94

D. Applications of Linear Equations

As a mathematical tool, linear equations rank among the most common, powerful, and versatile. In all cases, it’s important to remember that slope represents a rate of change.

The notation \( m = \frac{\Delta y}{\Delta x} \) literally means the quantity measured along the \( y \)-axis, is changing with respect to changes in the quantity measured along the \( x \)-axis.

EXAMPLE 11  Relating Temperature to Altitude

In meteorological studies, atmospheric temperature depends on the altitude according to the formula \( T(h) = -3.5h + 58.5 \), where \( T(h) \) represents the approximate Fahrenheit temperature at height \( h \) (in thousands of feet, \( 0 \leq h \leq 36 \)).

**a.** Interpret the meaning of the slope in this context.

**b.** Determine the temperature at an altitude of 12,000 ft.

**c.** If the temperature is \(-8^\circ\text{F}\) what is the approximate altitude?

**Algebraic Solution**

a. Notice that \( h \) is the input variable and \( T \) is the output. This shows \( \frac{\Delta T}{\Delta h} = \frac{-3.5}{1} \),

meaning the temperature drops 3.5°F for every 1000-ft increase in altitude.

b. Since height is in thousands, use \( h = 12 \).

\[
T(h) = -3.5h + 58.5
\]

original formula

\[
T(12) = -3.5(12) + 58.5
\]

substitute 12 for \( h \)

\[
= 16.5
\]

result

**Technology Solution**

\[
Y1(12) \quad 16.5
\]

At a height of 12,000 ft, the temperature is about 16.5°F.
c. Replacing \( T(h) \) with \(-8\) and solving gives

**Algebraic Solution**

\[
T(h) = -3.5h + 58.5 \quad \text{(original formula)}
\]

\[-8 = -3.5h + 58.5 \quad \text{(substitute \(-8\) for \(T(h)\))}
\]

\[-66.5 = -3.5h \quad \text{(subtract 58.5)}
\]

\[19 = h \quad \text{(divide by \(-3.5\))}
\]

The temperature is about \(-8^\circ\text{F}\) at a height of \(19 \times 1000 = 19,000\) ft.

**Graphical Solution**

Since we're given \(0 \leq h \leq 36\), we can set \(X_{\text{min}} = 0\) and \(X_{\text{max}} = 40\). At ground level \((x = 0)\), the formula gives a temperature of 58.5°C, while at \(h = 36\), we have \(T(36) = -67.5\). This shows appropriate settings for the range would be \(Y_{\text{min}} = -50\) and \(Y_{\text{max}} = 50\) (see figure). After setting \(Y_1 = -3.5X + 58.5\), we press \(\text{Tally}\) and move the cursor until we find an output value near \(-8\), which occurs when \(X\) is near 19. To check, we input 19 for \(x\) and the calculator displays an output of \(-8\), which corresponds with the algebraic result (at 19,000 ft, the temperature is \(-8^\circ\text{F}\)).

![Graph](image)

Now try Exercises 105 and 106

In many applications, outputs that are integer or rational values are rare, making it difficult to use the \(\text{Tally}\) feature alone to find an exact solution. In the Section 1.5, we'll develop additional ways that graphs and technology can be used to solve equations.

In some applications, the relationship is known to be linear but only a few points on the line are given. In this case, we can use two of the known data points to calculate the slope, then the point-slope form to find an equation model. One such application is linear depreciation, as when a government allows businesses to depreciate vehicles and equipment over time (the less a piece of equipment is worth, the less you pay in taxes).

**EXAMPLE 12A**

**Using Point-Slope Form to Find a Function Model**

Five years after purchase, the auditor of a newspaper company estimates the value of their printing press is $60,000. Eight years after its purchase, the value of the press had depreciated to $42,000. Find a linear equation that models this depreciation and discuss the slope and \(y\)-intercept in context.

**Solution**

Since the value of the press depends on time, the ordered pairs have the form \((t, v)\) where \(t\) is the input, and \(v\) is the output. This means the ordered pairs are \((5, 60,000)\) and \((8, 42,000)\).

\[
m = \frac{v_2 - v_1}{t_2 - t_1} \quad \text{slope formula}
\]

\[
= \frac{42,000 - 60,000}{8 - 5}
\]

\[
= \frac{-18,000}{3} = -6000
\]

\[
= \frac{1}{3}
\]

\[(t_1, v_1) = (5, 60,000); \quad (t_2, v_2) = (8, 42,000)\]
The slope of the line is \( \frac{\Delta \text{value}}{\Delta \text{time}} = -\frac{6000}{1} \), indicating the printing press loses \$6000\) in value with each passing year.

\[
\begin{align*}
v - v_1 &= m(t - t_1) & \text{point-slope form} \\
v - 60,000 &= -6000(t - 5) & \text{substitute } -6000 \text{ for } m; (5, 60,000) \text{ for } (t, v) \\
v - 60,000 &= -6000t + 30,000 & \text{simplify} \\
v &= -6000t + 90,000 & \text{solve for } v
\end{align*}
\]

The depreciation equation is \( v(t) = -6000t + 90,000 \). The \( v \)-intercept \((0, 90,000)\) indicates the original value (cost) of the equipment was \$90,000\).

Once the depreciation equation is found, it represents the (time, value) relationship for all future (and intermediate) ages of the press. In other words, we can now predict the value of the press for any given year. However, note that some equation models are valid for only a set period of time, and each model should be used with care.

**EXAMPLE 12B**

Using a Function Model to Gather Information

From Example 12A,

a. How much will the press be worth after 11 yr?

b. How many years until the value of the equipment is \$9000?\]

c. Is this function model valid for \( t = 18 \) yr (why or why not)?

**Solution**

a. Find the value \( v \) when \( t = 11 \):

\[
\begin{align*}
v(t) &= -6000t + 90,000 & \text{equation model} \\
v(11) &= -6000(11) + 90,000 & \text{substitute } 11 \text{ for } t \\
      &= 24,000 & \text{result } (11, 24,000)
\end{align*}
\]

After 11 yr, the printing press will only be worth \$24,000.

b. \( \ldots \text{value is } \$9000\) means \( v(t) = 9000 \):

\[
\begin{align*}
v(t) &= 9000 & \text{value at time } t \\
-6000t + 90,000 &= 9000 & \text{substitute } -6000t + 90,000 \text{ for } v(t) \\
-6000t &= -81,000 & \text{subtract } 90,000 \\
t &= 13.5 & \text{divide by } -6000
\end{align*}
\]

After 13.5 yr, the printing press will be worth \$9000.

c. Since substituting 18 for \( t \) gives a negative quantity, the function model is not valid for \( t = 18 \). In the current context, the model is only valid while \( v \geq 0 \) and solving \(-6000t + 90,000 \geq 0\) shows the domain of the function in this context is \( t \in [0, 15] \).

**Now try Exercises 107 through 112**
1.4 EXERCISES

CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For the equation \( y = \frac{-7}{4}x + 3 \), the slope is ________, and the y-intercept is ________.

2. The notation \( \frac{\Delta \text{cost}}{\Delta \text{time}} \) indicates the ________ is changing in response to changes in ________.

3. Line 1 has a slope of \(-0.4\). The slope of any line perpendicular to line 1 is ________.

4. The equation \( y - y_1 = m(x - x_1) \) is called the ________ form of a line.

5. Discuss/Explain how to graph a line using only the slope and a point on the line (no equations).

DEVELOPING YOUR SKILLS

Solve each equation for \( y \) and evaluate the result using \( x = -5, x = -2, x = 0, x = 1, \) and \( x = 3 \).

1. \( 4x + 5y = 10 \)  8. \( 3y - 2x = 9 \)
2. \( -0.4x + 0.2y = 1.4 \)  9. \( -0.2x + 0.7y = -2.1 \)
3. \( \frac{1}{3}x + \frac{1}{3}y = -1 \)  10. \( \frac{1}{3}y - \frac{1}{3}x = 2 \)

For each equation, solve for \( y \) and identify the new coefficient of \( x \) and new constant term.

13. \( 6x - 3y = 9 \)  14. \( 9y - 4x = 18 \)
15. \( -0.5x - 0.3y = 2.1 \)  16. \( -0.7x + 0.6y = -2.4 \)
17. \( \frac{2}{3}x + \frac{1}{2}y = -4 \)  18. \( \frac{7}{10}y - \frac{4}{15}x = \frac{7}{6} \)

Evaluate each equation by selecting three inputs that will result in integer values. Then graph each line.

19. \( y = -\frac{3}{4}x + 5 \)  20. \( y = \frac{3}{4}x + 1 \)
21. \( y = -\frac{3}{4}x - 2 \)  22. \( y = \frac{3}{2}x - 3 \)
23. \( y = -\frac{1}{3}x + 4 \)  24. \( y = -\frac{1}{3}x + 3 \)

Find the \( x \)- and \( y \)-intercepts for each line, then (a) use these two points to calculate the slope of the line, (b) write the equation with \( y \) in terms of \( x \) (solve for \( y \)) and (c) compare the calculated slope and \( y \)-intercept to the equation from part (b). Comment on what you notice.

25. \( 3x + 4y = 12 \)  26. \( 3y - 2x = -6 \)
27. \( 2x - 5y = 10 \)  28. \( 2x + 3y = 9 \)
29. \( 4x - 5y = -15 \)  30. \( 5y + 6x = -25 \)

Write each equation in slope-intercept form (solve for \( y \)) and function form, then identify the slope and \( y \)-intercept.

31. \( 2x + 3y = 6 \)  32. \( 4y - 3x = 12 \)
33. \( 5x + 4y = 20 \)  34. \( y + 2x = 4 \)
35. \( x = 3y \)  36. \( 2x = -5y \)
37. \( 3x + 4y - 12 = 0 \)  38. \( 5y - 3x + 20 = 0 \)

For Exercises 39 to 50, use the slope-intercept form to state the equation of each line. Verify your solutions to Exercises 45 to 47 using a graphing calculator.

39. \( y = \frac{1}{2}x + 2 \)  40. \( y = -\frac{3}{2}x + 3 \)

41. \( y = \frac{1}{2}x + 2 \)  42. \( m = -2; \ y \)-intercept \((0, -3)\)
43. \( m = 3; \ y \)-intercept \((0, 2)\)
44. \( m = -\frac{3}{2}; \ y \)-intercept \((0, -4)\)
45. \( m = -4; \ (-3, 2) \) is on the line
46. \( m = 2; (5, -3) \) is on the line
47. \( m = \frac{-3}{2}; (-4, 7) \) is on the line

48.

50.

Write each equation in slope-intercept form, then use the rate of change (slope) and \( y \)-intercept to graph the line.

51. \( 3x + 5y = 20 \)
52. \( 2y - x = 4 \)
53. \( 2x - 3y = 15 \)
54. \( -3x + 2y = 4 \)

Graph each linear equation using the \( y \)-intercept and rate of change (slope) determined from each equation.

55. \( y = \frac{3}{2}x + 3 \)
56. \( y = \frac{5}{2}x - 1 \)
57. \( y = -\frac{1}{3}x + 2 \)
58. \( y = \frac{\sqrt{2}}{3}x + 2 \)
59. \( y = 2x - 5 \)
60. \( y = -3x + 4 \)
61. \( y = \frac{1}{2}x - 3 \)
62. \( y = \frac{-3}{2}x + 2 \)

Find the equation of the line using the information given. Write answers in slope-intercept form.

63. parallel to \( 2x - 5y = 10 \), through the point \((-5, 2)\)
64. parallel to \( 6x + 9y = 27 \), through the point \((-3, -5)\)
65. perpendicular to \( 5y - 3x = 9 \), through the point \((6, 3)\)
66. perpendicular to \( x - 4y = 7 \), through the point \((-5, 3)\)
67. parallel to \( 12x + 5y = 65 \), through the point \((-2, -1)\)
68. parallel to \( 15y - 8x = 50 \), through the point \((3, -4)\)
69. parallel to \( y = -3 \), through the point \((2, 5)\)
70. perpendicular to \( y = -3 \) through the point \((2, 5)\)

Write the equations in slope-intercept form and state whether the lines are parallel, perpendicular, or neither.

71. \( 4y - 5x = 8 \)
72. \( 3y - 2x = 6 \)
5y + 4x = -15
2x + 3y = 6
73. \( 2x - 5y = 20 \)
74. \( -4x + 6y = 12 \)
4x - 3y = 18
2x + 3y = 6
75. \( 3x + 4y = 12 \)
76. \( 5y = 11x + 135 \)
6x + 8y = 2
11y + 5x = -77

A secant line is one that intersects a graph at two or more points. For each graph given, find an equation of the line (a) parallel and (b) perpendicular to the secant line, through the point indicated.

77.

78.

79.

80.

81.

82.

Find the equation of the line in point-slope form, then graph the line.

83. \( m = 2; P_1 = (2, -5) \)
84. \( m = -1; P_1 = (2, -3) \)
85. \( P_1 = (3, -4), P_2 = (11, -1) \)
86. \( P_1 = (-1, 6), P_2 = (5, 1) \)
87. \( m = 0.5; P_1 = (1.8, -3.1) \)
88. \( m = 1.5; P_1 = (-0.75, -0.125) \)
Find the equation of the line in point-slope form, and state the meaning of the slope in context—what information is the slope giving us?

89. 

90. 

91. 

92. 

93. 

94. 

Using the concept of slope, match each description with the graph that best illustrates it. Assume time is scaled on the horizontal axes, and height, speed, or distance from the origin (as the case may be) is scaled on the vertical axis.

95. While driving today, I got stopped by a state trooper. After she warned me to slow down, I continued on my way.

96. After hitting the ball, I began trotting around the bases shouting, "Ooh, ooh, ooh!" When I saw it wasn't a home run, I began sprinting.

97. At first I ran at a steady pace, then I got tired and walked the rest of the way.

98. While on my daily walk, I had to run for a while when I was chased by a stray dog.

99. I climbed up a tree, then I jumped out.

100. I steadily swam laps at the pool yesterday.

101. I walked toward the candy machine, stared at it for a while then changed my mind and walked back.

102. For practice, the girls' track team did a series of 25-m sprints, with a brief rest in between.

WORKING WITH FORMULAS

103. General linear equation: \( ax + by = c \)

The general equation of a line is shown here, where \( a, b, \) and \( c \) are real numbers, with \( a \) and \( b \) not simultaneously zero. Solve the equation for \( y \) and note the slope (coefficient of \( x \)) and \( y \)-intercept (constant term). Use these to find the slope and \( y \)-intercept of the following lines, without solving for \( y \) or computing points.

\[ \text{a. } 3x + 4y = 8 \quad \text{b. } 2x + 5y = -15 \]
\[ \text{c. } 5x - 6y = -12 \quad \text{d. } 3y - 5x = 9 \]

104. Intercept-Intercept form of a linear equation: \( \frac{x}{h} + \frac{y}{k} = 1 \)

The \( x \)- and \( y \)-intercepts of a line can also be found by writing the equation in the form shown (with the equation set equal to 1). The \( x \)-intercept will be \((h, 0)\) and the \( y \)-intercept will be \((0, k)\). Find the \( x \)- and \( y \)-intercepts of the following lines using this method. How is the slope of each line related to the values of \( h \) and \( k \)?

\[ \text{a. } 2x + 5y = 10 \quad \text{b. } 3x - 4y = -12 \]
\[ \text{c. } 5x + 4y = 8 \]
105. **Speed of sound:** The speed of sound as it travels through the air depends on the temperature of the air according to the function $V = \frac{4}{3}T + 331$, where $V$ represents the velocity of the sound waves in meters per second (m/s), at a temperature of $T^\circ$ Celsius. (a) Interpret the meaning of the slope and y-intercept in this context. (b) Determine the speed of sound at a temperature of $20^\circ$C. (c) If the speed of sound is measured at 361 m/s, what is the temperature of the air?

106. **Acceleration:** A driver going down a straight highway is traveling 60 ft/sec (about 41 mph) on cruise control, when he begins accelerating at a rate of 5.2 ft/sec$^2$. The final velocity of the car is given by $V = \frac{29}{5}t + 60$, where $V$ is the velocity at time $t$. (a) Interpret the meaning of the slope and y-intercept in this context. (b) Determine the velocity of the car after 9.4 seconds. (c) If the car is traveling at 100 ft/sec, for how long did it accelerate?

107. **Investing in coins:** The purchase of a “collector’s item” is often made in hopes the item will increase in value. In 1998, Mark purchased a 1909-S VDB Lincoln Cent (in fair condition) for $150. By the year 2004, its value had grown to $190. (a) Use the relation (time since purchase, value) with $t = 0$ corresponding to 1998 to find a linear equation modeling the value of the coin. (b) Discuss what the slope and y-intercept indicate in this context. (c) How much was the penny worth in 2009? (d) How many years after purchase will the penny’s value exceed $250? (e) If the penny is now worth $170, how many years has Mark owned the penny?

108. **Depreciation:** Once a piece of equipment is put into service, its value begins to depreciate. A business purchases some computer equipment for $18,500. At the end of a 2-yr period, the value of the equipment has decreased to $11,500. (a) Use the relation (time since purchase, value) to find a linear equation modeling the value of the equipment. (b) Discuss what the slope and y-intercept indicate in this context. (c) What is the equipment’s value after 4 yr? (d) How many years after purchase will the value decrease to $6000? (e) Generally, companies will sell used equipment while it still has value and use the funds to purchase new equipment. According to the function, how many years will it take this equipment to depreciate in value to $1000?

109. **Internet connections:** The number of households that are hooked up to the Internet (homes that are online) has been increasing steadily in recent years. In 1995, approximately 9 million homes were online. By 2001 this figure had climbed to about 51 million. (a) Use the relation (year, homes online) with $t = 0$ corresponding to 1995 to find an equation model for the number of homes online. (b) Discuss what the slope indicates in this context. (c) According to this model, in what year did the first homes begin to come online? (d) If the rate of change stays constant, how many households were on the Internet in 2006? (e) How many years after 1995 will there be over 100 million households connected? (f) If there are 115 million households connected, what year is it?

*Source: 2004 Statistical Abstract of the United States, Table 965*

110. **Prescription drugs:** Retail sales of prescription drugs have been increasing steadily in recent years. In 1995, retail sales hit $72 billion. By the year 2000, sales had grown to about $146 billion. (a) Use the relation (year, retail sales of prescription drugs) with $t = 0$ corresponding to 1995 to find a linear equation modeling the growth of retail sales. (b) Discuss what the slope indicates in this context. (c) According to this model, in what year will sales reach $250 billion? (d) According to the model, what was the value of retail prescription drug sales in 2005? (e) How many years after 1995 will retail sales exceed $279 billion? (f) If yearly sales totaled $294 billion, what year is it?

*Source: 2004 Statistical Abstract of the United States, Table 122*

111. **Prison population:** In 1990, the number of persons sentenced and serving time in state and federal institutions was approximately 740,000. By the year 2000, this figure had grown to nearly 1,320,000. (a) Find a linear function with $t = 0$ corresponding to 1990 that models this data, (b) discuss the slope ratio in context, and (c) use the equation to estimate the prison population in 2010 if this trend continues.

*Source: Bureau of Justice Statistics at www.ojp.usdoj.gov/bjs*

112. **Eating out:** In 1990, Americans bought an average of 143 meals per year at restaurants. This phenomenon continued to grow in popularity and in the year 2000, the average reached 170 meals per year. (a) Find a linear function with $t = 0$ corresponding to 1990 that models this growth, (b) discuss the slope ratio in context, and (c) use the equation to estimate the average number of times an American will eat at a restaurant in 2010 if the trend continues.

*Source: The NPD Group, Inc., National Eating Trends, 2002*
EXTENDING THE CONCEPT


114. The general form of a linear equation is $ax + by = c$, where $a$ and $b$ are not simultaneously zero. (a) Find the $x$- and $y$-intercepts using the general form (substitute 0 for $x$, then 0 for $y$). Based on what you see, when does the intercept method work most efficiently? (b) Find the slope and $y$-intercept using the general form (solve for $y$). Based on what you see, when does the slope-intercept method work most efficiently?

115. Match the correct graph to the conditions stated for $m$ and $b$. There are more choices than graphs.

a. $m < 0$, $b < 0$  
   b. $m > 0$, $b < 0$

b. $m < 0$, $b > 0$  
   c. $m > 0$, $b > 0$

b. $m = 0$, $b > 0$  
   d. $m > 0$, $b = 0$

g. $m > 0$, $b = 0$  
   h. $m = 0$, $b < 0$

MAINTAINING YOUR SKILLS

116. (1.3) Determine the domain:
   a. $y = \sqrt{2x - 5}$
   b. $y = \frac{5}{2x - 5}$

117. (Appendix A.6) Simply without the use of a calculator.
   a. $27\frac{3}{4}$  
   b. $\sqrt{81x^2}$

118. (Appendix A.3) Three equations follow. One is an identity, another is a contradiction, and a third has a solution. State which is which.

   $2(x - 5) + 13 - 1 = 9 - 7 + 2x$
   $2(x - 4) + 13 - 1 = 9 + 7 - 2x$
   $2(x - 5) + 13 - 1 = 9 + 7 + 2x$

119. (Appendix A.2) Compute the area of the circular sidewalk shown here ($A = \pi r^2$). Use your calculator's value of $\pi$ and round the answer (only) to hundredths.

[Diagram of a circular sidewalk with a radius of 10 yards and a central radius of 8 yards]
In this section, we’ll build on many of the ideas developed in Appendix A.3 (Solving Linear Equations and Inequalities), as we learn to manipulate formulas and employ certain problem-solving strategies. We will also extend our understanding of graphical solutions to a point where they can be applied to virtually any family of functions.

### A. Solving Equations Graphically Using the Intersect Method

For some background on why a graphical solution is effective, consider the equation \(2x - 9 = -3(x - 1) - 2\). By definition, an **equation** is a statement that two expressions are equal for some value of the variable (Appendix A.3). To highlight this fact, the expressions \(2x - 9\) and \(-3(x - 1) - 2\) are evaluated independently for selected integers in Tables 1.4 and 1.5.

#### Table 1.4

<table>
<thead>
<tr>
<th>(x)</th>
<th>(2x - 9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-15</td>
</tr>
<tr>
<td>-2</td>
<td>-13</td>
</tr>
<tr>
<td>-1</td>
<td>-11</td>
</tr>
<tr>
<td>0</td>
<td>-9</td>
</tr>
<tr>
<td>1</td>
<td>-7</td>
</tr>
<tr>
<td>2</td>
<td>-5</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
</tr>
</tbody>
</table>

#### Table 1.5

<table>
<thead>
<tr>
<th>(x)</th>
<th>(-3(x - 1) - 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>10</td>
</tr>
<tr>
<td>-2</td>
<td>7</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>-5</td>
</tr>
<tr>
<td>3</td>
<td>-8</td>
</tr>
</tbody>
</table>

Note the two expressions are equal (the equation is true) only when the input is \(x = 2\). Solving equations graphically is a simple extension of this observation. By treating the expression on the left as the independent function \(Y_1\), we have \(Y_1 = 2X - 9\) and the related linear graph will contain all ordered pairs shown in Table 1.4 (see Figure 1.67). Doing the same for the right-hand expression yields \(Y_2 = -3(X - 1) - 2\), and its related graph will likewise contain all ordered pairs shown in the Table 1.5 (see Figure 1.68).

\[
\frac{2x - 9}{Y_1} = \frac{-3(x - 1) - 2}{Y_2}
\]

The solution is then found where \(Y_1 = Y_2\), or in other words, at the point where these two lines intersect (if it exists). See Figure 1.69.

Most graphing calculators have an **Intersect** feature that can quickly find the point(s) where two graphs intersect. On many calculators, we access this ability using the sequence \(\text{Y=} \rightarrow \text{calc} \rightarrow \text{5:intersect}\) (CALC) and selecting option 5:Intersect (Figure 1.70).
Because the calculator can work with up to 10 expressions at once, it will ask you to identify each graph you want to work with—even when there are only two. A marker is displayed on each graph in turn, and named in the upper left corner of the window (Figure 1.71). You can select a graph by pressing or bypass a graph by pressing one of the arrow keys. For situations involving multiple graphs or multiple solutions, the calculator offers a “GUESS?” option that enables you to specify the approximate location of the solution you’re interested in (Figure 1.72). For now, we’ll simply press two times in succession to identify each graph, and a third time to bypass the “GUESS?” option. The calculator then finds and displays the point of intersection (Figure 1.73). Be sure to check the settings on your viewing window before you begin, and if the point of intersection is not visible, try 3:Zoom Out or other window-resizing features to help locate it.

**EXAMPLE 1A**

**Solving an Equation Graphically**

Solve the equation $2(x - 3) + 7 = \frac{1}{2}x - 2$ using a graphing calculator.

**Solution**

Begin by entering the left-hand expression as $Y_1$ and the right-hand expression as $Y_2$ (Figure 1.74). To find points of intersection, press (CALC) and select option 5:intersect, which automatically places you on the graphing window, and asks you to identify the “First curve?” As discussed, pressing three times in succession will identify each graph, bypass the “Guess?” option, then find and display the point of intersection (Figure 1.75). Here the point of intersection is $(-2, -3)$, showing the solution to this equation is $x = -2$ (for which both expressions equal $-3$). This can be verified by direct substitution or by using the TABLE feature.

This method of solving equations is called the **Intersection-of-Graphs** method, and can be applied to many different equation types.
**Intersection-of-Graphs Method for Solving Equations**

For any equation of the form \( f(x) = g(x) \),
1. Assign \( f(x) \) as \( Y_1 \) and \( g(x) \) as \( Y_2 \).
2. Graph both functions and identify any point(s) of intersection, if they exist.
   
The \( x \)-coordinate of all such points is a solution to the equation.

Recall that in the solution of linear equations, we sometimes encounter equations that are identities (infinitely many solutions) or contradictions (no solutions). These possibilities also have graphical representations, and appear as coincident lines and parallel lines respectively. These possibilities are illustrated in Figure 1.76.

![Figure 1.76](image)

One point of intersection (unique solution)

Infinitely many points of intersection (identity)

No point of intersection (contradiction)

---

**EXAMPLE 1B**

Solving an Equation Graphically

Solve \( 0.75x + 2 = 0.5(1 + 1.5x) - 3 \) using a graphing calculator.

**Solution**

With \( 0.75X + 2 \) as \( Y_1 \) and \( 0.5(1 + 1.5X) - 3 \) as \( Y_2 \), we use the \( \text{2nd} \) \( \text{TRACE} \) (CALC) option and select \( 5: \text{intersect} \). The graphs appear to be parallel lines (Figure 1.77), and after pressing \( \) three times we obtain the error message shown (Figure 1.78), confirming there are no solutions.

![Figure 1.77](image)

**Figure 1.77**

![Figure 1.78](image)

**Figure 1.78**

A. You've just seen how we can solve equations using the Intersection-of-Graphs method

---

**B. Solving Equations Graphically Using the x-Intercept/Zeroes Method**

The intersection-of-graphs method works extremely well when the graphs of \( f(x) \) and \( g(x) \) \( (Y_1 \) and \( Y_2) \) are simple and "well-behaved." Later in this course, we encounter a number of graphs that are more complex, and it will help to develop alternative methods for solving graphically. Recall that two equations are equivalent if they have the same solution set. For instance, the equations \( 2x = 6 \) and \( 2x - 6 = 0 \) are equivalent (since \( x = 3 \) is a solution to both), as are \( 3x - 1 = x + 5 \) and \( 2x - 6 = 0 \) (since...
Section 1.5 Solving Equations and Inequalities Graphically; Formulas and Problem Solving

$x = 3$ is a solution to both. Applying the intersection-of-graphs method to the last two equivalent equations, gives

\[
\begin{align*}
3x - 1 &= x + 5 \quad &Y_1 \\
2x - 6 &= 0 \quad &Y_2
\end{align*}
\]

The intersection method will work equally well in both cases, but the equation on the right has only one variable expression, and will produce a single (visible) graph (since \(Y_2 = 0\) is simply the \(x\)-axis). Note that here we seek an input value that will result in an output of 0. In other words, all solutions will have the form \((x, 0)\), which is the \(x\)-intercept of the graph. For this reason, the method is alternatively called the zeroes method or the \(x\)-intercept method. The method employs the approach shown above, in which the equation \(f(x) = g(x)\) is rewritten as \(f(x) - g(x) = 0\), with \(f(x) - g(x)\) assigned as \(Y_1\).

**Zeroes/x-Intercept Method for Solving Equations**

For any equation of the form \(f(x) = g(x)\),

1. Rewrite the equation as \(f(x) - g(x) = 0\).
2. Assign \(f(x) - g(x)\) as \(Y_1\).
3. Graph the resulting function and identify any \(x\)-intercepts, if they exist.

Any \(x\)-intercept(s) of the graph will be a solution to the equation.

To locate the zero (\(x\)-intercept) for \(2x - 6 = 0\) on a graphing calculator, enter \(2x - 6\) for \(Y_1\) and use the \(2\):zero option found on the same menu as the \(5\):intersect option (Figure 1.79). Since some equations have more than one zero, the \(2\):zero option will ask you to "narrow down" the interval it should search, even though there is only one zero here. It does this by asking for a "Left Bound?", a "Right Bound?", and a "GUESS?" (the Guess? option can once again be bypassed). You can enter these bounds by tracing along the graph or by inputting a chosen value, then pressing \(\text{Graph}\) (note how the calculator posts a marker at each bound). Figure 1.80 shows we entered \(x = 0\) as the left bound and \(x = 4\) as the right, and the calculator will search for the \(x\)-intercept in this interval (note that in general, the cursor will be either above or below the \(x\)-axis for the left bound, but must be on the opposite side of the \(x\)-axis for the right bound). Pressing \(\text{Graph}\) once more bypasses the Guess? option and locates the \(x\)-intercept at \((3, 0)\). The solution is \(x = 3\) (Figure 1.81).
EXAMPLE 2  ▶ Solving an Equation Using the Zeroes Method

Solve the equation $-4(x + 3) - 6 = 2x + 3$ using the zeroes method.

**Solution ▶**

As given, we have $f(x) = -4(x + 3) - 6$ and $g(x) = 2x + 3$. Rewriting the equation as $f(x) - g(x) = 0$ gives $-4(x + 3) - 6 - (2x + 3) = 0$, where the expression for $g(x)$ is parenthesized to ensure the equations remain equivalent. Entering $-4(X + 3) - 6 - (2X + 3)$ as $Y_1$ and pressing $\text{CALC} \ 2: \text{zero}$ produces the screen shown in Figure 1.82, with the calculator requesting a left bound. We can input any $x$-value that is obviously to the left of the $x$-intercept, or move the cursor to any position left of the $x$-intercept and press $\square$ (we input $x = -4$, see Figure 1.83). The calculator then asks for a right bound and as before we can input any $x$-value obviously to the right, or simply move the cursor to any location on the opposite side of the $x$-axis and press $\square$ (we chose $x = -2$, see Figure 1.84). After bypassing the $\text{Guess?}$ option (press $\square$ once again), the calculator locates the $x$-intercept at $(-3.5, 0)$, and the solution to the original equation is $x = -3.5$ (Figure 1.85).

B. You've just seen how we can solve equations using the $x$-intercept/zeroes method

C. Solving Linear Inequalities Graphically

The intersection-of-graphs method can also be applied to solve linear inequalities. The point of intersection simply becomes one of the boundary points for the solution interval, and is included or excluded depending on the inequality given. For the inequality $f(x) > g(x)$ written as $Y_1 > Y_2$, it becomes clear the inequality is true for all inputs $x$ where the outputs for $Y_1$ are greater than the outputs for $Y_2$, meaning the graph of $f(x)$ is above the graph of $g(x)$. A similar statement can be made for $f(x) < g(x)$ written as $Y_1 < Y_2$.

**Intersection-of-Graphs Method for Solving Inequalities**

For any inequality of the form $f(x) > g(x)$,
1. Assign $f(x)$ as $Y_1$ and $g(x)$ as $Y_2$.
2. Graph both functions and identify any point(s) of intersection, if they exist.

The solution set is all real numbers $x$ for which the graph of $Y_1$ is above the graph of $Y_2$.

For strict inequalities, the boundary of the solution interval is not included. A similar process is used for the inequalities $f(x) \geq g(x)$, $f(x) < g(x)$, and $f(x) \leq g(x)$. Note that we can actually draw the graphs of $Y_1$ and $Y_2$ differently (one more bold than the
EXAMPLE 3  

Solving an Inequality Using the Intersection-of-Graphs Method

Solve \(0.5(3 - x) - 5 \leq 2x + 4\) using the intersection-of-graphs method.

Solution

To assist with the clarity of the solution, we set the calculator to graph \(Y_2\) using a bolder line than \(Y_1\) (Figure 1.86). With \(0.5(3 - x) - 5\) as \(Y_1\) and \(2x + 4\) as \(Y_2\), we use the \(\text{CALC}\) option and select \(5:\text{intersect}\). Pressing \(\text{ENTER}\) three times serves to identify both graphs, bypass the “Guess?” option, and display the point of intersection \((-3, -2)\) (Figure 1.87). Since the graph of \(Y_1\) is below the graph of \(Y_2\) \((Y_1 \leq Y_2)\) for all values of \(x\) to the right of \((-3, -2)\), \(x = -3\) is the left boundary, with the interval extending to positive infinity.

Due to the less than or equal to inequality, we include \(x = -3\) and the solution interval is \(x \in [-3, \infty)\).

\[\text{Now try Exercises 27 through 36}\]

D. Solving for a Specified Variable in Literal Equations

A formula is an equation that models a known relationship between two or more quantities. A literal equation is simply one that has two or more variables. Formulas are a type of literal equation, but not every literal equation is a formula. For example, the formula \(A = P + PRT\) models the growth of money in an account earning simple interest, where \(A\) represents the total amount accumulated, \(P\) is the initial deposit, \(R\) is the annual interest rate, and \(T\) is the number of years the money is left on deposit. To describe \(A = P + PRT\), we might say the formula has been “solved for \(A\)” or that “\(A\) is written in terms of \(P\), \(R\), and \(T\)”.

In some cases, before using a formula it may be convenient to solve for one of the other variables, say \(P\). In this case, \(P\) is called the object variable.

EXAMPLE 4  

Solving for Specified Variable

Given \(A = P + PRT\), write \(P\) in terms of \(A\), \(R\), and \(T\) (solve for \(P\)).

Solution

Since the object variable occurs in more than one term, we first apply the distributive property.

\[A = P + PRT\]

\[A = P(1 + RT)\]

\[\frac{A}{1 + RT} = P\]

\[\text{Now try Exercises 37 through 48}\]
We solve literal equations for a specified variable using the same methods we used for other equations and formulas. Remember that it’s good practice to focus on the object variable to help guide you through the solution process, as again shown in Example 5.

**EXAMPLE 5**

**Solving for a Specified Variable**

Given $2x + 3y = 15$, write $y$ in terms of $x$ (solve for $y$).

**Solution**

$2x + 3y = 15$

$\frac{1}{3}(3y) = \frac{1}{3}(-2x + 15)$

$y = -\frac{2}{3}x + 5$

**WORTHY OF NOTE**

In Example 5, notice that in the second step we wrote the subtraction of $2x$ as $-2x + 15$ instead of $15 - 2x$. For reasons that will become clearer as we continue our study, we generally write variable terms before constant terms.

**Now try Exercises 49 through 54**

**Literal Equations and General Solutions**

Solving literal equations for a specified variable can help us develop the general solution for an entire family of equations. This is demonstrated here for the family of linear equations written in the form $ax + b = c$. A side-by-side comparison with a specific linear equation demonstrates that identical ideas are used.

<table>
<thead>
<tr>
<th>Specific Equation</th>
<th>Literal Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x + 3 = 15$</td>
<td>focus on variable</td>
</tr>
<tr>
<td>$2x = 15 - 3$</td>
<td>subtract constant</td>
</tr>
<tr>
<td>$x = \frac{15 - 3}{2}$</td>
<td>divide by coefficient</td>
</tr>
<tr>
<td>$ax + b = c$</td>
<td>$ax = c - b$</td>
</tr>
<tr>
<td>$x = \frac{c - b}{a}$</td>
<td>$x = -\frac{b}{a}$</td>
</tr>
</tbody>
</table>

Of course the solution on the left would be written as $x = 6$ and checked in the original equation. On the right we now have a general formula for all equations of the form $ax + b = c$.

**EXAMPLE 6**

**Solving Equations of the Form $ax + b = c$ Using a General Formula**

Solve $6x - 1 = -25$ using the formula just developed, and check your solution in the original equation.

**Solution**

For this equation, $a = 6$, $b = -1$, and $c = -25$, giving

$x = \frac{c - b}{a}$

$= \frac{-25 - (-1)}{6}$

$= \frac{-24}{6}$

$= -4$

→ Check: $6x - 1 = -25$

$6(-4) - 1 = -25$

$-24 - 1 = -25$

$-25 = -25 \checkmark$

**Now try Exercises 55 through 60**

**A. You've just seen how we can solve for a specified variable in a formula or literal equation**

Developing a general solution for the linear equation $ax + b = c$ seems to have little practical use. But in Section 3.2 we'll use this idea to develop a general solution for quadratic equations, a result with much greater significance.
E. Using a Problem-Solving Guide

Becoming a good problem solver is an evolutionary process. Over time and with continued effort, your problem-solving skills grow, as will your ability to solve a wider range of applications. Most good problem solvers develop the following characteristics:

- A positive attitude
- A mastery of basic facts
- Strong mental arithmetic skills
- Good mental-visual skills
- Good estimation skills
- A willingness to persevere

These characteristics form a solid basis for applying what we call the Problem-Solving Guide, which simply organizes the basic elements of good problem solving. Using this guide will help save you from two common stumbling blocks—indecision and not knowing where to start.

Problem-Solving Guide

- **Gather and organize information.**
  Read the problem several times, forming a mental picture as you read. Highlight key phrases. List given information, including any related formulas. Clearly identify what you are asked to find.

- **Make the problem visual.**
  Draw and label a diagram or create a table of values, as appropriate. This will help you see how different parts of the problem fit together.

- **Develop an equation model.**
  Assign a variable to represent what you are asked to find and build any related expressions referred to in the problem. Write an equation model based on the relationships given in the problem. Carefully reread the problem to double-check your equation model.

- **Use the model and given information to solve the problem.**
  Substitute given values, then simplify and solve. State the answer in sentence form, and check that the answer is reasonable. Include any units of measure indicated.

General Modeling Exercises

Translating word phrases into symbols is an important part of building equations from information given in paragraph form. Sometimes the variable occurs more than once in the equation, because two different items in the same exercise are related. If the relationship involves a comparison of size, we often use line segments or bar graphs to model the relative sizes.

**EXAMPLE 7** - Solving an Application Using the Problem-Solving Guide

The largest state in the United States is Alaska (AK), which covers an area that is 230 square miles (mi²) more than 500 times that of the smallest state, Rhode Island (RI). If they have a combined area of 616,460 mi², how many square miles does each cover?

**Solution** - Combined area is 616,460 mi², AK covers 230 more than 500 times the area of RI.

- Gather and organize information
- Highlight any key phrases
- Make the problem visual
Let \( R \) represent the area of Rhode Island. Then \( 500R + 230 \) represents Alaska’s area.

\[
\text{Rhode Island's area + Alaska’s area = Total}
\]

\[
R + (500R + 230) = 616,460
\]

\[
501R = 616,230
\]

\[
R = 1230
\]

Rhode Island covers an area of 1230 mi\(^2\), while Alaska covers an area of 500(1230) + 230 = 615,230 mi\(^2\).

**Consecutive Integer Exercises**

Exercises involving **consecutive integers** offer excellent practice in assigning variables to unknown quantities, building related expressions, and the problem-solving process in general. We sometimes work with consecutive **odd** integers or consecutive **even** integers as well.

**EXAMPLE 8**

**Solving a Problem Involving Consecutive Odd Integers**

The sum of three consecutive **odd** integers is 69. What are the integers?

**Solution**

The sum of three consecutive odd integers . . .

![Diagram showing three consecutive odd integers with labels (n, n+2, n+4)]

Let \( n \) represent the smallest consecutive odd integer, then \( n + 2 \) represents the second odd integer and \( n + 4 \) represents the third.

In words: first + second + third odd integer = 69

\[
n + (n + 2) + (n + 4) = 69
\]

\[
3n + 6 = 69
\]

\[
3n = 63
\]

\[
n = 21
\]

The odd integers are \( n = 21 \), \( n + 2 = 23 \), and \( n + 4 = 25 \).

\[21 + 23 + 25 = 69 \checkmark\]

**Uniform Motion (Distance, Rate, Time) Exercises**

Uniform motion problems have many variations, and it’s important to draw a good diagram when you get started. Recall that if speed is constant, the distance traveled is equal to the rate of speed multiplied by the time in motion: \( D = RT \).
EXAMPLE 9  ►  Solving a Problem Involving Uniform Motion

I live 260 mi from a popular mountain retreat. On my way there to do some mountain biking, my car had engine trouble—forcing me to bike the rest of the way. If I drove 2 hr longer than I biked and averaged 60 miles per hour driving and 10 miles per hour biking, how many hours did I spend pedaling to the resort?

Solution  ►  The sum of the two distances must be 260 mi. The rates are given, and the driving time is 2 hr more than biking time.

<table>
<thead>
<tr>
<th>Driving</th>
<th>Biking</th>
</tr>
</thead>
<tbody>
<tr>
<td>Home</td>
<td>Resort</td>
</tr>
</tbody>
</table>

\[ D_1 = RT \quad D_2 = rt \]

\[ D_1 + D_2 = \text{Total distance} \]

260 miles

Let \( t \) represent the biking time, then \( T = t + 2 \) represents time spent driving.

\[ D_1 + D_2 = 260 \]

\[ RT + rt = 260 \]

\[ 60(t + 2) + 10t = 260 \]

\[ 70t + 120 = 260 \]

\[ 70t = 140 \]

\[ t = 2 \]

I rode my bike for \( t = 2 \) hr, after driving \( t + 2 = 4 \) hr.

Now try Exercises 73 through 76

Exercises Involving Mixtures

Mixture problems offer another opportunity to refine our problem-solving skills while using many elements from the problem-solving guide. They also lend themselves to a very useful mental-visual image and have many practical applications.

EXAMPLE 10  ►  Solving an Application Involving Mixtures

As a nasal decongestant, doctors sometimes prescribe saline solutions with a concentration between 6% and 20%. In "the old days," pharmacists had to create different mixtures, but only needed to stock these concentrations, since any percentage in between could be obtained using a mixture. An order comes in for a 15% solution. How many milliliters (mL) of the 20% solution must be mixed with 10 mL of the 6% solution to obtain the desired 15% solution? Provide both an algebraic solution and a graphical solution.
**Algebraic Solution**

Only 6% and 20% concentrations are available; mix some 20% solution with 10 mL of the 6% solution. (See Figure 1.88.)

![Figure 1.88](image)

Let $x$ represent the amount of 20% solution, then $10 + x$ represents the total amount of 15% solution.

<table>
<thead>
<tr>
<th>1st quantity times</th>
<th>2nd quantity times</th>
<th>1st+2nd quantity times</th>
</tr>
</thead>
<tbody>
<tr>
<td>its concentration</td>
<td>its concentration</td>
<td>desired concentration</td>
</tr>
<tr>
<td>$10(0.06)$</td>
<td>$x(0.2)$</td>
<td>$(10 + x)(0.15)$</td>
</tr>
<tr>
<td>$0.6$</td>
<td>$0.2x$</td>
<td>$1.5 + 0.15x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.2x = 0.9 + 0.15x$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$0.05x = 0.9$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x = 18$</td>
</tr>
</tbody>
</table>

To obtain a 15% solution, 18 mL of the 20% solution must be mixed with 10 mL of the 6% solution.

**Graphical Solution**

**WORTHY OF NOTE**

For mixture exercises, an estimate assuming equal amounts of each liquid can be helpful. For example, assume we use 10 mL of the 8% solution and 10 mL of the 20% solution. The final concentration would be halfway in between, $\frac{8+20}{2} = 13\%$. This is too low a concentration (we need a 15% solution), so we know that more than 10 mL of the stronger (20%) solution must be used.

Although both methods work equally well, here we elect to use the intersection-of-graphs method and enter $10(0.06) + x(0.2)$ as $Y_1$ and $(10 + x)(0.15)$ as $Y_2$. Virtually all graphical solutions require a careful study of the context to set the viewing window prior to graphing. If 10 mL of liquid were used from each concentration, we would have 20 mL of a 13% solution (see **Worthy of Note**), so more of the stronger solution is needed. This shows that an appropriate $X_{\text{max}}$ might be close to 30. If all 30 mL were used, the output would be $30(0.15) = 4.5$, so an appropriate $Y_{\text{max}}$ might be around 6 (see Figure 1.89). Using [GRAPH] [TRACE] (CALC) 5:Intersect and pressing three times gives $(18, 4.2)$ as the point of intersection, showing $x = 18$ mL of the stronger solution must be used (Figure 1.90).

---

**E.** You’ve just seen how we can use the problem-solving guide to solve various problem types

Now try Exercises 77 through 84
### CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. When using the ________ method, one side of an equation is entered as ___ and the other side as ___ on a graphing calculator. The ________ of the point of ________ is the solution of the equation.

2. To solve a linear inequality using the intersection-of-graphs method, first find the point of ________. The ________ of this point is a boundary value of the solution interval and if the inequality is not strict, this value is ________ in the solution.

3. A(n) ________ equation is an equation having ____) or more unknowns.

4. For the equation \( S = 2\pi r^2 + 2\pi rh \), we can say that \( S \) is written in terms of ____ and ____.

5. Discuss/Explain the similarities and differences between the intersection and zeroes methods for solving equations. How can the zeroes method be applied to solving linear inequalities? Give examples in your discussion.

6. Discuss/Explain each of the four basic parts of the problem-solving guide. Include a solved example in your discussion.

### DEVELOPING YOUR SKILLS

#### Solve the following equations using a graphing calculator and the intersection-of-graphs method. For Exercises 7 and 8, carefully sketch the graphs you designate as \( Y_1 \) and \( Y_2 \) by hand before using your calculator.

7. \( 3x - 7 = -2(x + 1) + 10 \)
8. \( -2x + 1 = 2(x - 3) - 1 \)
9. \( 0.8x + 0.4 = 0.25(2 - 0.4x) - 2.8 \)
10. \( 0.5x + 2.5 = 0.75(3 - 0.2x) - 0.4 \)
11. \( x - (3x + 1) = -0.5(4x + 6) + 2 \)
12. \( 3x - (4 + x) = 0.2(6 + 10x) - 5.2 \)
13. \( \frac{1}{3}x = \frac{2}{3}x - 9 \)
14. \( \frac{-2}{5}x + 8 = \frac{2}{5}x \)
15. \( \frac{1}{2}(x - 4) + 10 = x - (2 + \frac{1}{2}x) \)
16. \( \frac{-1}{3}(x + 6) + 5 = -x - (6 - \frac{2}{3}x) \)

#### Solve the following equations using a graphing calculator and the \( x \)-intercept/zeroes method. Compare your results for Exercises 17 and 18 to those of Exercises 7 and 8.

17. \( 3x - 7 = -2(x + 1) + 10 \)
18. \( -2x + 1 = 2(x - 3) - 1 \)
19. \( 2(3 - 2x) - 5 = -3x + 3 \)
20. \( -3(-3 - x) + 4 = 2x + 5 \)
21. \( -1.5(x + 4) + 2.5 = 3x - 3.5 \)
22. \( 0.8(3x - 1) + 0.2 = -2x + 3.8 \)
23. \( 2(x + 2) + 1 = x + (1 + x) \)
24. \( 3(2x - 1) - 1 = 2x - (1 - 4x) \)
25. \( 3x - (0.7x - 1.2) = 2(1.1x + 0.6) + 0.1x \)
26. \( 3x + 2(0.2x - 1.4) = 4(0.8x - 0.7) + 0.2x \)
27. \( 3x - 7 > -2(x + 1) + 10 \)
28. \( -2x + 1 > 2(x - 3) - 1 \)
29. \( 2x - (3 + x) \geq -2(5 - 2x) + 7 \)
30. \( 4(3x - 5) + 2 \geq 3(2 - 4x) + 24 \)
31. \( -0.3(x + 2) + 1.1 < 0.2x + 3 \)
32. \( 0.25(4 - x) + 1 < 1 - 0.5x \)
33. \( -3(x - 1) + 1 \leq x - 4(x - 1) \)
34. \( 1.1(2 - x) + 0.2 > 5(0.1 - 0.2x) - 0.1x \)
35. \( 2(1.5x - 1.1) + 0.1x \geq 4x - 0.3(3x - 4) \)
36. \( 4(x - 1) - 2x + 7 < 2(x + 1.5) \)
Solve for the specified variable in each formula or literal equation.

37. \( P = C + CM \) for \( C \) (retail)
38. \( S = P - PD \) for \( P \) (retail)
39. \( C = 2\pi r \) for \( r \) (geometry)
40. \( V = LWH \) for \( W \) (geometry)
41. \( \frac{P_1V_1}{T_1} = \frac{P_2V_2}{T_2} \) for \( T_2 \) (science)
42. \( \frac{C}{P_2} = \frac{P_1}{d^2} \) for \( P_2 \) (communication)
43. \( V = \frac{2}{3}\pi r^2h \) for \( h \) (geometry)
44. \( V = \frac{1}{3}\pi r^2h \) for \( h \) (geometry)
45. \( S_n = n\left(\frac{a_1 + a_n}{2}\right) \) for \( n \) (sequences)
46. \( A = \frac{h(b_1 + b_2)}{2} \) for \( h \) (geometry)
47. \( S = B + \frac{1}{2}PS \) for \( P \) (geometry)

48. \( s = \frac{1}{2}gt^2 + vt \) for \( g \) (physics)
49. \( Ax + By = C \) for \( y \)
50. \( 2x + 3y = 6 \) for \( y \)
51. \( \frac{2}{3}x + \frac{3}{2}y = 2 \) for \( y \)
52. \( \frac{3}{2}x - \frac{2}{3}y = 12 \) for \( y \)
53. \( y - 3 = \frac{4}{3}(x + 10) \) for \( y \)
54. \( y + 4 = \frac{12}{15}(x + 10) \) for \( y \)

The following equations are given in \( ax + b = c \) form. Solve by identifying the value of \( a \), \( b \), and \( c \), then using the formula \( x = \frac{c - b}{a} \).

55. \( 3x + 2 = -19 \)
56. \( 7x + 5 = 47 \)
57. \( -6x + 1 = 33 \)
58. \( -4x + 9 = 43 \)
59. \( 7x - 13 = -27 \)
60. \( 3x - 4 = -25 \)

WORKING WITH FORMULAS

61. Surface area of a cylinder: \( SA = 2\pi r^2 + 2\pi rh \)
   The surface area of a cylinder is given by the formula shown, where \( h \) is the height of the cylinder and \( r \) is the radius of the base. Find the height of a cylinder that has a radius of 8 cm and a surface area of 1256 cm². Use \( \pi \approx 3.14 \).

62. Using the equation-solving process for Exercise 61 as a model, solve the formula \( SA = 2\pi r^2 + 2\pi rh \) for \( h \).

APPLICATIONS

Solve by building an equation model and using the problem-solving guidelines as needed. Check all answers using a graphing calculator.

General Modeling Exercises

63. Two spelunkers (cave explorers) were exploring different branches of an underground cavern. The first was able to descend 198 ft farther than twice the second. If the first spelunker descended 1218 ft, how far was the second spelunker able to descend?

64. The area near the joining of the Tigris and Euphrates Rivers (in modern Iraq) has often been called the Cradle of Civilization, since the area has evidence of many ancient cultures. The length of the Euphrates River exceeds that of the Tigris by 620 mi. If they have a combined length of 2880 mi, how long is each river?

65. U.S. postal regulations require that a package can have a maximum combined length and girth (distance around) of 108 in. A shipping carton is constructed so that it has a width of 14 in., a height of 12 in., and can be cut or folded to various lengths. What is the maximum length that can be used?

Source: www.USPS.com
66. Hi-Tech Home Improvements buys a fleet of identical trucks that cost $32,750 each. The company is allowed to depreciate the value of their trucks for tax purposes by $5250 per year. If company policies dictate that older trucks must be sold once their value declines to $6500, approximately how many years will they keep these trucks?

67. The longest suspension bridge in the world is the Akashi Kaikyo (Japan) with a length of 6532 feet. Japan is also home to the Shimotsui Straight bridge. The Akashi Kaikyo bridge is 364 ft more than twice the length of the Shimotsui bridge. How long is the Shimotsui bridge?

Source: www.guinessworldrecords.com

68. The Mars rover Spirit landed on January 3, 2004. Just over 1 yr later, on January 14, 2005, the Huygens probe landed on Titan (one of Saturn’s moons). At their closest approach, the distance from the Earth to Saturn is 29 million mi more than 21 times the distance from the Earth to Mars. If the distance to Saturn is 743 million mi, what is the distance to Mars?

Consecutive Integer Exercises

69. Find two consecutive even integers such that the sum of twice the smaller integer plus the larger integer is one hundred forty-six.

70. When the smaller of two consecutive integers is added to three times the larger, the result is fifty-one. Find the smaller integer.

71. Seven times the first of two consecutive odd integers is equal to five times the second. Find each integer.

72. Find three consecutive even integers where the sum of triple the first and twice the second is eight more than four times the third.

Uniform Motion Exercises

73. At 9:00 A.M., Linda leaves work on a business trip, gets on the interstate, and sets her cruise control at 60 mph. At 9:30 A.M., Bruce notices she’s left her briefcase and cell phone, and immediately starts after her driving 75 mph. At what time will Bruce catch up with Linda?

74. A plane flying at 300 mph has a 3-hr head start on a “chase plane,” which has a speed of 800 mph. How far from the airport will the chase plane overtake the first plane?

75. Jeff had a job interview in a nearby city 72 mi away. On the first leg of the trip he drove an average of 30 mph through a long construction zone, but was able to drive 60 mph after passing through this zone. If driving time for the trip was 1 1/2 hr, how long was he driving in the construction zone?

76. At a high-school cross-country meet, Jared jogged 8 mph for the first part of the race, then increased his speed to 12 mph for the second part. If the race was 21 mi long and Jared finished in 2 hr, how far did he jog at the faster pace?

Mixture Exercises

Give the total amount of the mix that results and the percent concentration or worth of the mix.

77. Two quarts of 100% orange juice are mixed with 2 quarts of water (0% juice).

78. Ten pints of a 40% acid are combined with 10 pints of an 80% acid.

79. Eight pounds of premium coffee beans worth $2.50 per pound are mixed with 8 lb of standard beans worth $1.10 per pound.

80. A rancher mixes 50 lb of a custom feed blend costing $1.80 per pound, with 50 lb of cheap cottonseed worth $0.60 per pound.

Solve each application of the mixture concept.

81. To help sell more of a lower grade meat, a butcher mixes some premium ground beef worth $3.10/lb, with 8 lb of lower grade ground beef worth $2.05/lb. If the result was an intermediate grade of ground beef worth $2.68/lb, how much premium ground beef was used?
82. Knowing that the camping/hiking season has arrived, a nutrition outlet is mixing GORP (Good Old Raisins and Peanuts) for the anticipated customers. How many pounds of peanuts worth $1.29/lb, should be mixed with 20 lb of deluxe raisins worth $1.89/lb, to obtain a mix that will sell for $1.49/lb?

83. How many pounds of walnuts at 84¢/lb should be mixed with 20 lb of pecans at $1.20/lb to give a mixture worth $1.04/lb?

84. How many pounds of cheese worth 81¢/lb must be mixed with 10 lb cheese worth $1.29/lb to make a mixture worth $1.11/lb?

EXTENDING THE CONCEPT

85. Look up and read the following article. Then turn in a one page summary. "Don’t Give Up!,” William H. Kraus, Mathematics Teacher, Volume 86, Number 2, February 1993: pages 110–112.

86. A chemist has four solutions of a very rare and expensive chemical that are 15% acid (cost $120 per ounce), 20% acid (cost $180 per ounce), 35% acid (cost $280 per ounce) and 45% acid (cost $359 per ounce). She requires 200 oz of a 29% acid solution. Find the combination of any two of these concentrations that will minimize the total cost of the mix.

87. \[ P, Q, R, S, T, \text{ and } U \text{ represent numbers. The arrows in the figure show the sum of the two or three numbers added in the indicated direction.} \]

MAINTAINING YOUR SKILLS

89. (Appendix A.5) Solve for \( x \):
   \[
   \frac{1}{x + 2} - \frac{2}{x} = \frac{3}{x^2 + 2x}
   \]

90. (1.4) Solve for \( y \), then state the slope and \( y \)-intercept of the line: \(-6x + 7y = 42\)

91. (Appendix A.4) Factor each expression:
   a. \( 4x^2 - 9 \)
   b. \( x^3 - 27 \)

92. (1.3) Given \( g(x) = x^2 - 3x - 10 \), evaluate \( g\left(\frac{1}{2}\right), g(-2), \text{ and } g(5) \)

(Example: \( Q + T = 23 \). Find \( P + Q + R + S + T + U \).)

88. Given a sphere circumscribed by a cylinder, verify the volume of the sphere is \( \frac{2}{3} \) that of the cylinder.
1.6 Linear Function Models and Real Data

LEARNING OBJECTIVES

In Section 1.6 you will see how we can:

☐ A. Draw a scatterplot and identify positive and negative associations

☐ B. Use a scatterplot to identify linear and nonlinear associations

☐ C. Use a scatterplot to identify strong and weak correlations

☐ D. Find a linear function that models the relationships observed in a set of data

☐ E. Use linear regression to find the line of best fit

Collecting and analyzing data is a tremendously important mathematical endeavor, having applications throughout business, industry, science, and government. The link between classroom mathematics and real-world mathematics is called a regression, in which we attempt to find an equation that will act as a model for the raw data. In this section, we focus on situations where the data is best modeled by a linear function.

A. Scatterplots and Positive/Negative Associations

In this section, we continue our study of ordered pairs and functions, but this time using data collected from various sources or from observed real-world relationships. You can hardly pick up a newspaper or magazine without noticing it contains a large volume of data presented in graphs, charts, and tables. In addition, there are many simple experiments or activities that enable you to collect your own data. We begin analyzing the collected data using a scatterplot, which is simply a graph of all of the ordered pairs in a data set. Often, real data (sometimes called raw data) is not very "well behaved" and the points may be somewhat scattered—the reason for the name.

Positive and Negative Associations

Earlier we noted that lines with positive slope rise from left to right, while lines with negative slope fall from left to right. We can extend this idea to the data from a scatterplot. The data points in Example 1A seem to rise as you move from left to right, with larger input values generally resulting in larger outputs. In this case, we say there is a positive association between the variables. If the data seems to decrease or fall as you move left to right, we say there is a negative association.

EXAMPLE 1A

Drawing a Scatterplot and Observing Associations

The ratio of the federal debt to the total population is known as the per capita debt. The per capita debt of the United States is shown in the table for the odd-numbered years from 1997 to 2007. Draw a scatterplot of the data and state whether the association is positive, negative, or cannot be determined.

Source: Data from the Bureau of Public Debt at www.publicdebt.treas.gov

<table>
<thead>
<tr>
<th>Year</th>
<th>Per Capita Debt ($1000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>20.0</td>
</tr>
<tr>
<td>1999</td>
<td>20.7</td>
</tr>
<tr>
<td>2001</td>
<td>20.5</td>
</tr>
<tr>
<td>2003</td>
<td>23.3</td>
</tr>
<tr>
<td>2005</td>
<td>27.6</td>
</tr>
<tr>
<td>2007</td>
<td>30.4</td>
</tr>
</tbody>
</table>

Solution

Since the amount of debt depends on the year, year is the input x and per capita debt is the output y. Scale the x-axis from 1997 to 2007 and the y-axis from 20 to 30 to comfortably fit the data (the "squiggly lines," near the 20 and 1997 in the graph are used to show that some initial values have been skipped). The graph indicates a positive association between the variables, meaning the debt is generally increasing as time goes on.
EXAMPLE 1B • Drawing a Scatterplot and Observing Associations

A cup of coffee is placed on a table and allowed to cool. The temperature of the coffee is measured every 10 min and the data are shown in the table. Draw the scatterplot and state whether the association is positive, negative, or cannot be determined.

<table>
<thead>
<tr>
<th>Elapsed Time (minutes)</th>
<th>Temperature (°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>110</td>
</tr>
<tr>
<td>10</td>
<td>89</td>
</tr>
<tr>
<td>20</td>
<td>76</td>
</tr>
<tr>
<td>30</td>
<td>72</td>
</tr>
<tr>
<td>40</td>
<td>71</td>
</tr>
</tbody>
</table>

Solution

Since temperature depends on cooling time, time is the input $x$ and temperature is the output $y$. Scale the $x$-axis from 0 to 40 and the $y$-axis from 70 to 110 to comfortably fit the data. As you see in the figure, there is a negative association between the variables, meaning the temperature decreases over time.

A. You've just seen how we can draw a scatterplot and identify positive and negative associations

B. Scatterplots and Linear/Nonlinear Associations

The data in Example 1A had a positive association, while the association in Example 1B was negative. But the data from these examples differ in another important way. In Example 1A, the data seem to cluster about an imaginary line. This indicates a linear equation model might be a good approximation for the data, and we say there is a linear association between the variables. The data in Example 1B could not accurately be modeled using a straight line, and we say the variables time and cooling temperature exhibit a nonlinear association.

EXAMPLE 2 • Drawing a Scatterplot and Observing Associations

A college professor tracked her annual salary for 2002 to 2009 and the data are shown in the table. Draw the scatterplot and determine if there is a linear or nonlinear association between the variables. Also state whether the association is positive, negative, or cannot be determined.

<table>
<thead>
<tr>
<th>Year</th>
<th>Salary ($1000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2002</td>
<td>30.5</td>
</tr>
<tr>
<td>2003</td>
<td>31</td>
</tr>
<tr>
<td>2004</td>
<td>32</td>
</tr>
<tr>
<td>2005</td>
<td>33.2</td>
</tr>
<tr>
<td>2006</td>
<td>35.5</td>
</tr>
<tr>
<td>2007</td>
<td>39.5</td>
</tr>
<tr>
<td>2008</td>
<td>45.5</td>
</tr>
<tr>
<td>2009</td>
<td>52</td>
</tr>
</tbody>
</table>

Now try Exercises 7 and 8
Solution

Since salary earned depends on a given year, \textit{year} is the input \( x \) and \textit{salary} is the output \( y \). Scale the \( x \)-axis from 2002 to 2010, and the \( y \)-axis from 30 to 55 to comfortably fit the data. A line doesn’t seem to model the data very well, and the association appears to be nonlinear. The data rises from left to right, indicating a positive association between the variables. This makes good sense, since we expect our salaries to increase over time.

C. Identifying Strong and Weak Correlations

Using Figures 1.91 and 1.92 shown, we can make one additional observation regarding the data in a scatterplot. While both associations shown appear linear, the data in Figure 1.91 seems to cluster more tightly about an imaginary straight line than the data in Figure 1.92.

![Figure 1.91](image1)

![Figure 1.92](image2)

We refer to this “clustering” as the “goodness of fit,” or in statistical terms, the strength of the correlation. To quantify this fit we use a measure called the correlation coefficient \( r \), which tells whether the association is positive or negative: \( r > 0 \) or \( r < 0 \), and quantifies the strength of the association: \( |r| \leq 100\% \). Actually, the coefficient is given in decimal form, making \( |r| \leq 1 \). If the data points form a perfectly straight line, we say the strength of the correlation is either \(-1\) or \(1\), depending on the association. If the data points appear clustered about the line, but are scattered on either side of it, the strength of the correlation falls somewhere between \(-1\) and \(1\), depending on how tightly/loosely they’re scattered. This is summarized in Figure 1.93.

![Figure 1.93](image3)

The following scatterplots help to further illustrate this idea. Figure 1.94 shows a linear and negative association between the value of a car and the age of a car, with a strong correlation. Figure 1.95 shows there is no apparent association between family income and the number of children, and Figure 1.96 appears to show a linear and positive association between a man’s height and weight, with a weak correlation.
Until we develop a more accurate method of calculating a numerical value for this correlation, the best we can do are these broad generalizations: weak correlation, strong correlation, or no correlation.

EXAMPLE 3A  High School and College GPAs

Many colleges use a student’s high school GPA as a possible indication of their future college GPA. Use the data from Table 1.6 (high school/college GPA) to draw a scatterplot. Then

a. Sketch a line that seems to approximate the data, meaning it has the same general direction while passing through the observed “center” of the data.

b. State whether the association is positive, negative, or cannot be determined.

c. Decide whether the correlation is weak or strong.

<table>
<thead>
<tr>
<th>High School GPA</th>
<th>College GPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8</td>
<td>1.8</td>
</tr>
<tr>
<td>2.2</td>
<td>2.3</td>
</tr>
<tr>
<td>2.8</td>
<td>2.5</td>
</tr>
<tr>
<td>3.2</td>
<td>2.9</td>
</tr>
<tr>
<td>3.4</td>
<td>3.6</td>
</tr>
<tr>
<td>3.8</td>
<td>3.9</td>
</tr>
</tbody>
</table>

Solution

a. A line approximating the data set as a whole is shown in the figure.

b. Since the line has positive slope, there is a positive association between a student’s high school GPA and their GPA in college.

c. The correlation appears strong.

EXAMPLE 3B  Natural Gas Consumption

The amount of natural gas consumed by homes and offices varies with the season, with the highest consumption occurring in the winter months. Use the data from Table 1.7 (outdoor temperature/gas consumed) to draw a scatterplot. Then

a. Sketch an estimated line of best fit.

b. State whether the association is positive, negative, or cannot be determined.

c. Decide whether the correlation is weak or strong.
### Solution

a. We again use appropriate scales and sketch a line that seems to model the data (see figure).

b. There is a negative association between temperature and the amount of natural gas consumed.

c. The correlation appears to be strong.

D. Linear Functions That Model Relationships Observed in a Set of Data

Finding a linear function model for a set of data involves visually estimating and sketching a line that appears to best "fit" the data. This means answers will vary slightly, but a good, usable model can often be obtained. To find the function, we select two points on this imaginary line and use either the slope-intercept form or the point-slope formula to construct the function. Points on this estimated line but not actually in the data set can still be used to help determine the function.

#### Example 4

Finding a Linear Function to Model the Relationship Between GPAs

Use the scatterplot from Example 3A to find a function model for the line a college might use to project an incoming student's future GPA.

**Solution**

Any two points on or near the estimated best-fit line can be used to help determine the linear function (see the figure in Example 3A). For the slope, it's best to pick two points that are some distance apart, as this tends to improve the accuracy of the model. It appears (1.8, 1.8) and (3.8, 3.9) are both on the line, giving

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3.9 - 1.8}{3.8 - 1.8} = \frac{1.05}{2} = 0.525.
\]

Using the point-slope form with (1.8, 1.8),

\[
y - 1.8 = 0.525(x - 1.8),
\]

and solving for \(y\),

\[
y = 0.525x - 0.915.
\]

One possible function model for this data is \(f(x) = 0.525x - 0.915\). Slightly different functions may be obtained, depending on the points chosen.
The function from Example 4 predicts that a student with a high school GPA of 3.2 will have a college GPA of almost 3.3; \( f(3.2) = 1.05(3.2) - 0.09 = 3.3 \), yet the data gives an actual value of only 2.9. When working with data and function models, we should expect some variation when the two are compared, especially if the correlation is weak.

Applications of data analysis can be found in virtually all fields of study. In Example 5 we apply these ideas to an Olympic swimming event.

**EXAMPLE 5**

**Finding a Linear Function to Model the Relationship (Year, Gold Medal Times)**

The men’s 400-m freestyle times (gold medal times—to the nearest second) for the 1976 through 2008 Olympics are given in Table 1.8 (1900→0). Let the year be the input \( x \), and winning race time be the output \( y \). Based on the data, draw a scatterplot and answer the following questions.

- Does the association appear linear or nonlinear?
- Is the association positive or negative?
- Classify the correlation as weak or strong.
- Find a function model that approximates the data, then use it to predict the winning time for the 2012 Olympics.

<table>
<thead>
<tr>
<th>Year ( (x) ) ((1900→0))</th>
<th>Time ( (y) ) (\text{sec})</th>
</tr>
</thead>
<tbody>
<tr>
<td>76</td>
<td>232</td>
</tr>
<tr>
<td>80</td>
<td>231</td>
</tr>
<tr>
<td>84</td>
<td>231</td>
</tr>
<tr>
<td>88</td>
<td>227</td>
</tr>
<tr>
<td>92</td>
<td>225</td>
</tr>
<tr>
<td>96</td>
<td>228</td>
</tr>
<tr>
<td>100</td>
<td>221</td>
</tr>
<tr>
<td>104</td>
<td>223</td>
</tr>
<tr>
<td>108</td>
<td>223</td>
</tr>
</tbody>
</table>

**Solution**

Begin by choosing appropriate scales for the axes. The \( x \)-axis (year) could be scaled from 76 to 112, and the \( y \)-axis (swim time) from 210 to 246. This will allow for a “frame” around the data. After plotting the points, we obtain the scatterplot shown in the figure.

- The association appears to be linear.
- The association is negative, showing that finishing times tend to decrease over time.
- There is a moderate to strong correlation.
- The points \((76, 232)\) and \((104, 223)\) appear to be on a line approximating the data, and we select these to develop our equation model.

\[
\begin{align*}
    m &= \frac{y_2 - y_1}{x_2 - x_1} \\
    &= \frac{223 - 232}{104 - 76} \\
    &= -0.32 \\
    \text{slope formula} \\
    y - 232 &= -0.32(x - 76) \\
    \text{point-slope form} \\
    y - 232 &= -0.32x + 24.32 \\
    \text{distribute} \\
    y &= -0.32x + 256.32 \\
    \text{add 232 (solve for y)}
\end{align*}
\]
One model for this data is \( y = -0.32x + 256.32 \). Based on this model, the predicted time for the 2012 Olympics would be

\[
f(x) = -0.32x + 256.32 \\
\text{function model}
\]

\[
f(112) = -0.32(112) + 256.32 \\
\text{substitute 112 for } x \text{ (2012)}
\]

\[
= 220.48
\text{result}
\]

In 2012 the winning time is projected to be about 220.5 sec.

As a reminder, great care should be taken when using equation models obtained from real data. It would be foolish to assume that in the year 2700, swim times for the 400-m freestyle would be near 0 sec—even though that’s what the model predicts for \( x = 800 \). Most function models are limited by numerous constraining factors, and data collected over a much longer period of time might even be better approximated using a nonlinear model.

### E. Linear Regression and the Line of Best Fit

There is actually a sophisticated method for calculating the equation of a line that best fits a data set, called the **regression line**. The method minimizes the vertical distance between all data points and the line itself, making it the unique **line of best fit**. Most graphing calculators have the ability to perform this calculation quickly. The process involves these steps: (1) clearing old data, (2) entering new data, (3) displaying the data, (4) calculating the regression line, and (5) displaying and using the regression line. We’ll illustrate by finding the regression line for the data shown in Table 1.8 in Example 5, which gives the men’s 400-m freestyle gold medal times (in seconds) for the 1976 through the 2008 Olympics, with 1900→0.

#### Step 1: Clear Old Data

To prepare for the new data, we first clear out any old data. Press the \( \text{\textasciicircum} \) key and select option 4:ClrList. This places the ClrList command on the home screen. We tell the calculator which lists to clear by pressing \( \text{\textasciicircum} \) 1 to indicate List1 (L1), then enter a comma using the \( \text{\textasciicircum} \) key, and continue entering other lists we want to clear: \( \text{\textasciicircum} \) 2 \( \text{\textasciicircum} \) 3 will clear List1 (L1), List2 (L2), and List3 (L3).

#### Step 2: Enter New Data

Press the \( \text{\textasciicircum} \) key and select option 1:Edit. Move the cursor to the first position of List1, and simply enter the data from the first column of Table 1.8 in order: 76, 80, 84, and so on. Then use the right arrow \( \text{\textasciicircum} \) to navigate to List2, and enter the data from the second column: 232, 231, 231, and so on. When finished, you should obtain the screen shown in Figure 1.97.

#### Step 3: Display the Data

With the data held in these lists, we can now display the related ordered pairs on the coordinate grid. First press the \( \text{\textasciicircum} \) key and \( \text{\textasciicircum} \) any existing equations. Then press \( \text{\textasciicircum} \) \( \text{\textasciicircum} \) to access the "STATPLOTS" screen. With the cursor on \( \text{\textasciicircum} \) Plot1, press \( \text{\textasciicircum} \) and be sure the options shown in Figure 1.98 are highlighted. If you need to make any changes, navigate the cursor to

---

**WORTHY OF NOTE**

As a rule of thumb, the tick marks for \( X\text{max} \) can be set by mentally estimating \( \frac{X\text{max} + X\text{min}}{2} \) and using a convenient number in the neighborhood of the result (the same goes for \( Y\text{max} \)). As an alternative to manually setting the window, the \( \text{\textasciicircum} \) \( \text{\textasciicircum} \) ZoomStat feature can be used.
the desired option and press \( \text{Graph} \). Note the data in L1 ranges from 76 to 108, while the data in L2 ranges from 221 to 232. This means an appropriate viewing window might be \([70, 120]\) for the \( x \)-values, and \([210, 240]\) for the \( y \)-values. Press the \( \text{Graph} \) key and set up the window accordingly. After you’re finished, pressing the \( \text{Graph} \) key should produce the graph shown in Figure 1.99.

**Step 4: Calculate the Regression Equation**

To have the calculator compute the regression equation, press the \( \text{Math} \) and \( \text{Calc} \) keys to move the cursor over to the \text{CALC} options (see Figure 1.100). Since it appears the data is best modeled by a linear equation, we choose option \( 4: \text{LinReg(ax + b)} \). Pressing the number 4 places this option on the home screen, and pressing \( \text{Graph} \) computes the values of \( a \) and \( b \) (the calculator automatically uses the values in L1 and L2 unless instructed otherwise). Rounded to hundredths, the linear regression model is \( y = -0.33x + 257.06 \) (Figure 1.101).

**Step 5: Display and Use the Results**

Although graphing calculators have the ability to paste the regression equation directly into \( Y_1 \) on the \( \text{Y=} \) screen, for now we’ll enter \( Y_1 = -0.33x + 257.06 \) by hand. Afterward, pressing the \( \text{Graph} \) key will plot the data points (if Plot1 is still active) and graph the line. Your display screen should now look like the one in Figure 1.102. The regression line is the best estimator for the set of data as a whole, but there will still be some difference between the values it generates and the values from the set of raw data (the output in Figure 1.102 shows the estimated time for the 2000 Olympics was about 224 sec, when actually it was the year Ian Thorpe of Australia set a world record of 221 sec).

---

**EXAMPLE 6**

**Using Regression to Model Employee Performance**

Riverside Electronics reviews employee performance semiannually, and awards increases in their hourly rate of pay based on the review. The table shows Thomas’ hourly wage for the last 4 yr (eight reviews). Find the regression equation for the data and use it to project his hourly wage for the year 2011, after his fourteenth review.

<table>
<thead>
<tr>
<th>Review (x)</th>
<th>Wage (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2004) 1</td>
<td>$9.58</td>
</tr>
<tr>
<td>2</td>
<td>$9.75</td>
</tr>
<tr>
<td>(2005) 3</td>
<td>$10.54</td>
</tr>
<tr>
<td>4</td>
<td>$11.41</td>
</tr>
<tr>
<td>(2006) 5</td>
<td>$11.60</td>
</tr>
<tr>
<td>6</td>
<td>$11.91</td>
</tr>
<tr>
<td>(2007) 7</td>
<td>$12.11</td>
</tr>
<tr>
<td>8</td>
<td>$13.02</td>
</tr>
</tbody>
</table>

**Solution**

Following the prescribed sequence produces the equation \( y = 0.48x + 9.09 \). For \( x = 14 \) we obtain \( y = 0.48(14) + 9.09 \) or a wage of $15.81. According to this model, Thomas will be earning $15.81 per hour in 2011.
With each linear regression, the calculator can be set to compute a correlation coefficient that is a measure of how well the equation fits the data (see Subsection C). To display this "r-value" use \[\text{2nd} \ 0\] (CATALOG) and activate DiagnosticOn. Figure 1.103 shows a scatterplot with perfect negative correlation \((r = -1)\) and notice all data points are on the line. Figure 1.104 shows a strong positive correlation \((r \approx 0.98)\) of the data from Example 6. See Exercise 35.

![Figure 1.103](image1)

![Figure 1.104](image2)

E. You've just seen how we can use a linear regression to find the line of best fit

---

### 1.6 EXERCISES

#### CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. When the ordered pairs from a set of data are plotted on a coordinate grid, the result is called a _____.

2. If the data points seem to form a curved pattern or if no pattern is apparent, the data is said to have a _____ association.

3. If the data points seem to cluster along an imaginary line, the data is said to have a _____ association.

4. If the pattern of data points seems to increase as they are viewed left to right, the data is said to have a _____ association.

5. Compare/Contrast: One scatterplot is linear, with a weak and positive association. Another is linear, with a strong and negative association. Give a written description of each scatterplot.

6. Discuss/Explain how this is possible: Working from the same scatterplot, Demetrius obtained the equation \(y = -0.64x + 44\) for his equation model, while Jessie got the equation \(y = -0.59x + 42\).
DEVELOPING YOUR SKILLS

7. For mail with a high priority, "Express Mail" offers next day delivery by 12:00 noon to most destinations, 365 days of the year. The service was first offered by the U.S. Postal Service in the early 1980s and has been growing in use ever since. The cost of the service (in cents) for selected years is shown in the table. (a) Draw a scatterplot of the data, then (b) decide if the association is positive, negative, or cannot be determined.

<table>
<thead>
<tr>
<th>Year</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1981</td>
<td>935</td>
</tr>
<tr>
<td>1985</td>
<td>1075</td>
</tr>
<tr>
<td>1988</td>
<td>1200</td>
</tr>
<tr>
<td>1991</td>
<td>1395</td>
</tr>
<tr>
<td>1995</td>
<td>1500</td>
</tr>
<tr>
<td>1999</td>
<td>1575</td>
</tr>
<tr>
<td>2002</td>
<td>1785</td>
</tr>
<tr>
<td>2010</td>
<td>1830</td>
</tr>
</tbody>
</table>

The volume is shown in the table for 2002, and the odd numbered years from 1991 to 2001 (in billions of shares).

(a) Draw a scatterplot of the data, (b) decide if the association is linear or nonlinear, and (c) if the association is positive, negative, or cannot be determined.

Source: 2004 Statistical Abstract of the United States, Table 1202

8. After the Surgeon General's first warning in 1964, cigarette consumption began a steady decline as advertising was banned from television and radio, and public awareness of the dangers of cigarette smoking grew. The percentage of the U.S. adult population who considered themselves smokers is shown in the table for selected years.

(a) Draw a scatterplot of the data, then (b) decide if the association is positive, negative, or cannot be determined.

<table>
<thead>
<tr>
<th>Year</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>1965</td>
<td>42.4</td>
</tr>
<tr>
<td>1974</td>
<td>37.1</td>
</tr>
<tr>
<td>1979</td>
<td>33.5</td>
</tr>
<tr>
<td>1985</td>
<td>29.9</td>
</tr>
<tr>
<td>1990</td>
<td>25.3</td>
</tr>
<tr>
<td>1995</td>
<td>24.6</td>
</tr>
<tr>
<td>2000</td>
<td>23.1</td>
</tr>
<tr>
<td>2002</td>
<td>22.4</td>
</tr>
<tr>
<td>2005</td>
<td>16.9</td>
</tr>
</tbody>
</table>


9. Since the 1970s women have made tremendous gains in the political arena, with more and more female candidates running for, and winning seats in the U.S. Senate and U.S. Congress. The number of women candidates for the U.S. Congress is shown in the table for selected years.

(a) Draw a scatterplot of the data, (b) decide if the association is linear or nonlinear and (c) if the association is positive, negative, or cannot be determined.

Source: Center for American Women and Politics at www.cawp.rutgers.edu/Facts3.html

<table>
<thead>
<tr>
<th>Year</th>
<th>Candidates</th>
</tr>
</thead>
<tbody>
<tr>
<td>1972</td>
<td>32</td>
</tr>
<tr>
<td>1978</td>
<td>46</td>
</tr>
<tr>
<td>1984</td>
<td>65</td>
</tr>
<tr>
<td>1992</td>
<td>106</td>
</tr>
<tr>
<td>1998</td>
<td>121</td>
</tr>
<tr>
<td>2004</td>
<td>141</td>
</tr>
</tbody>
</table>

10. The number of shares traded on the New York Stock Exchange experienced dramatic change in the 1990s as more and more individual investors gained access to the stock market via the Internet and online brokerage houses. The volume is shown in the table for 2002, and the odd numbered years from 1991 to 2001 (in billions of shares).

(a) Draw a scatterplot of the data, (b) decide if the association is linear or nonlinear, and (c) if the association is positive, negative, or cannot be determined.

<table>
<thead>
<tr>
<th>Year</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1991</td>
<td>46</td>
</tr>
<tr>
<td>1993</td>
<td>67</td>
</tr>
<tr>
<td>1995</td>
<td>88</td>
</tr>
<tr>
<td>1997</td>
<td>134</td>
</tr>
<tr>
<td>1999</td>
<td>206</td>
</tr>
<tr>
<td>2001</td>
<td>311</td>
</tr>
<tr>
<td>2002</td>
<td>369</td>
</tr>
</tbody>
</table>

Source: 2000 and 2004 Statistical Abstract of the United States, Table 1202

The data sets in Exercises 11 and 12 are known to be linear.

11. The total value of the goods and services produced by a nation is called its gross domestic product or GDP. The GDP per capita is the ratio of the GDP for a given year to the population that year, and is one of many indicators of economic health. The GDP per capita (in $1000s) for the United States is shown in the table for selected years.

(a) Draw a scatterplot using scales that appropriately fit the data, then sketch an estimated line of best fit, (b) decide if the association is positive or negative, then (c) decide whether the correlation is weak or strong.

<table>
<thead>
<tr>
<th>Year</th>
<th>GDP per Capita</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>5.1</td>
</tr>
<tr>
<td>1980</td>
<td>7.6</td>
</tr>
<tr>
<td>1990</td>
<td>12.3</td>
</tr>
<tr>
<td>2000</td>
<td>17.7</td>
</tr>
<tr>
<td>2010</td>
<td>23.3</td>
</tr>
<tr>
<td>2020</td>
<td>27.7</td>
</tr>
<tr>
<td>2030</td>
<td>35.0</td>
</tr>
<tr>
<td>2040</td>
<td>37.8</td>
</tr>
</tbody>
</table>

Source: 2004 Statistical Abstract of the United States, Tables 2 and 641

12. Real estate brokers carefully track sales of new homes looking for trends in location, price, size, and other factors. The table relates the average selling price within a price range (homes in the $120,000 to $140,000 range are represented by the $130,000 figure), to the number of new homes sold by Homestead Realty in 2004.

(a) Draw a scatterplot using scales that appropriately fit the data, then sketch an estimated line of best fit, (b) decide if the association is positive or negative, then (c) decide whether the correlation is weak or strong.

<table>
<thead>
<tr>
<th>Price</th>
<th>Sales</th>
</tr>
</thead>
<tbody>
<tr>
<td>130's</td>
<td>126</td>
</tr>
<tr>
<td>150's</td>
<td>95</td>
</tr>
<tr>
<td>170's</td>
<td>103</td>
</tr>
<tr>
<td>190's</td>
<td>75</td>
</tr>
<tr>
<td>210's</td>
<td>44</td>
</tr>
<tr>
<td>230's</td>
<td>59</td>
</tr>
<tr>
<td>250's</td>
<td>21</td>
</tr>
</tbody>
</table>
For the scatterplots given: (a) Arrange them in order from the weakest to the strongest correlation, (b) sketch a line that seems to approximate the data, (c) state whether the association is positive, negative, or cannot be determined, and (d) choose two points on (or near) the line and use them to approximate its slope (rounded to one decimal place).

13. A.  

14. A.  

15.  

16.  

17.  

18.  

19.  

20.  

21. In most areas of the country, law enforcement has become a major concern. The number of law enforcement officers employed by the federal government and having the authority to carry firearms and make arrests is shown in the table for selected years.

(a) Draw a scatterplot using scales that appropriately fit the data and sketch an estimated line of best fit and (b) decide if the association is positive or negative. (c) Choose two points on or near the estimated line of best fit, and use them to find a function model and predict the number of federal law enforcement officers in 1995 and the projected number for 2011. Answers may vary.


22. Due to atmospheric pressure, the temperature at which water will boil varies predictably with the altitude. Using special equipment designed to duplicate atmospheric pressure, a lab experiment is set up to study this relationship for altitudes up to 8000 ft. The set of data collected is shown in the table, with the boiling temperature \( y \) in degrees Fahrenheit, depending on the altitude \( x \) in feet. (a) Draw a scatterplot using scales that appropriately fit the data and sketch an estimated line of best fit, (b) decide if the association is positive or negative. (c) Choose two points on or near the estimated line of best fit, and use them to find a function model and predict the boiling point of water on the summit of Mt. Hood in Washington State (11,239 ft height), and along the shore of the Dead Sea (approximately 1312 ft below sea level). Answers may vary.

23. For the data given in Exercise 11 (Gross Domestic Product per Capita), choose two points on or near the line you sketched and use them to find a function model for the data. Based on this model, what is the projected GDP per capita for the year 2010?
24. For the data given in Exercise 12 (Sales by Real Estate Brokers), choose two points on or near the line you sketched and use them to find a function model for the data. Based on this model, how many sales can be expected for homes costing $275,000? $300,000?

WORKING WITH FORMULAS

25. Circumference of a Circle: \( C = 2\pi r \): The formula for the circumference of a circle can be written as a function of \( r \): \( C(r) = 2\pi r \). (a) Set up a table of values for \( r = 1 \) through 6 and draw a scatterplot of the data. (b) Is the association positive or negative? Why? (c) What can you say about the strength of the correlation? (d) Sketch a line that “approximates” the data. What can you say about the slope of this line?

26. Volume of a Cylinder: \( V = \pi r^2 h \): As part of a project, students cut a long piece of PVC pipe with a diameter of 10 cm into sections that are 5, 10, 15, 20, and 25 cm long. The bottom of each is then made watertight and each section is filled to the brim with water. The volume is then measured using a flask marked in cm³ and the results collected into the table shown. (a) Draw a scatterplot of the data. (b) Is the association positive or negative? Why? (c) What can you say about the strength of the correlation? (d) Would the correlation here be stronger or weaker than the correlation in Exercise 25? Why? (e) Run a linear regression to verify your response.

APPLICATONS

Use the regression capabilities of a graphing calculator to complete Exercises 27 through 34.

27. Height versus wingspan: Leonardo da Vinci’s famous diagram is an illustration of how the human body comes in predictable proportions. One such comparison is a person’s height to their wingspan (the maximum distance from the outstretched tip of one middle finger to the other). Careful measurements were taken on eight students and the set of data is shown here. Using the data, (a) draw the scatterplot; (b) determine whether the association is linear or nonlinear; (c) determine whether the association is positive or negative; and (d) find the regression equation and use it to predict the wingspan of a student with a height of 65 in.

<table>
<thead>
<tr>
<th>Height (x)</th>
<th>Wingspan (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>61</td>
<td>60.5</td>
</tr>
<tr>
<td>61.5</td>
<td>62.5</td>
</tr>
<tr>
<td>54.5</td>
<td>54.5</td>
</tr>
<tr>
<td>73</td>
<td>71.5</td>
</tr>
<tr>
<td>67.5</td>
<td>66</td>
</tr>
<tr>
<td>51</td>
<td>50.75</td>
</tr>
<tr>
<td>57.5</td>
<td>54</td>
</tr>
<tr>
<td>52</td>
<td>51.5</td>
</tr>
</tbody>
</table>

28. Patent applications: Every year the U.S. Patent and Trademark Office (USPTO) receives thousands of applications from scientists and inventors. The table given shows the number of applications received for the odd years from 1993 to 2003 (1990→0). Use the data to (a) draw the scatterplot; (b) determine whether the association is linear or nonlinear; (c) determine whether the association is positive or negative; and (d) find the regression equation and use it to predict the number of applications that will be received in 2011.

<table>
<thead>
<tr>
<th>Year (1990→0)</th>
<th>Applications (1000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>188.0</td>
</tr>
<tr>
<td>5</td>
<td>236.7</td>
</tr>
<tr>
<td>7</td>
<td>237.0</td>
</tr>
<tr>
<td>9</td>
<td>278.3</td>
</tr>
<tr>
<td>11</td>
<td>344.7</td>
</tr>
<tr>
<td>13</td>
<td>355.4</td>
</tr>
</tbody>
</table>

29. **Patents issued:** An increase in the number of patent applications (see Exercise 28), typically brings an increase in the number of patents issued, though many applications are denied due to improper filing, lack of scientific support, and other reasons. The table given shows the number of patents issued for the odd years from 1993 to 2003 (1999 → 0). Use the data to (a) draw the scatterplot; (b) determine whether the association is linear or nonlinear; (c) determine whether the association is positive or negative; and (d) find the regression equation and use it to predict the number of applications that will be approved in 2011. Which is increasing faster, the number of patent applications or the number of patents issued? How can you tell for sure?

*Source: United States Patent and Trademark Office at www.uspto.gov/web*

<table>
<thead>
<tr>
<th>Year (1990→0)</th>
<th>Patents (1000s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>107.3</td>
</tr>
<tr>
<td>5</td>
<td>114.2</td>
</tr>
<tr>
<td>7</td>
<td>122.9</td>
</tr>
<tr>
<td>9</td>
<td>159.2</td>
</tr>
<tr>
<td>11</td>
<td>187.8</td>
</tr>
<tr>
<td>13</td>
<td>189.6</td>
</tr>
</tbody>
</table>

31. **Females/males in the workforce:** Over the last 4 decades, the percentage of the female population in the workforce has been increasing at a fairly steady rate. At the same time, the percentage of the male population in the workforce has been declining. The set of data is shown in the tables. Using the data, (a) draw scatterplots for both data sets, (b) determine whether the associations are linear or nonlinear, (c) determine whether the associations are positive or negative, and (d) determine if the percentage of females in the workforce is increasing faster than the percentage of males is decreasing. Discuss/Explain how you can tell for sure.

*Source: 1998 Wall Street Journal Almanac, p. 316*

<table>
<thead>
<tr>
<th>Exercise 31 (women)</th>
<th>Exercise 31 (men)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Year (x) (1950→0)</td>
<td>Percent</td>
</tr>
<tr>
<td>5</td>
<td>36</td>
</tr>
<tr>
<td>10</td>
<td>38</td>
</tr>
<tr>
<td>15</td>
<td>39</td>
</tr>
<tr>
<td>20</td>
<td>43</td>
</tr>
<tr>
<td>25</td>
<td>46</td>
</tr>
<tr>
<td>30</td>
<td>52</td>
</tr>
<tr>
<td>35</td>
<td>55</td>
</tr>
<tr>
<td>40</td>
<td>58</td>
</tr>
<tr>
<td>45</td>
<td>59</td>
</tr>
<tr>
<td>50</td>
<td>60</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Year (x) (1950→0)</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>85</td>
</tr>
<tr>
<td>10</td>
<td>83</td>
</tr>
<tr>
<td>15</td>
<td>81</td>
</tr>
<tr>
<td>20</td>
<td>80</td>
</tr>
<tr>
<td>25</td>
<td>78</td>
</tr>
<tr>
<td>30</td>
<td>77</td>
</tr>
<tr>
<td>35</td>
<td>76</td>
</tr>
<tr>
<td>40</td>
<td>76</td>
</tr>
<tr>
<td>45</td>
<td>75</td>
</tr>
<tr>
<td>50</td>
<td>73</td>
</tr>
</tbody>
</table>

30. **High jump records:** In the sport of track and field, the high jumper is an unusual athlete. They seem to defy gravity as they launch their bodies over the high bar. The winning height at the summer Olympics (to the nearest unit) has steadily increased over time, as shown in the table for selected years. Using the data, (a) draw the scatterplot, (b) determine whether the association is linear or nonlinear, (c) determine whether the association is positive or negative, and (d) find the regression equation using \( t = 0 \) corresponding to 1900 and predict the winning height for the 2004 and 2008 Olympics. How close did the model come to the actual heights?

*Source: athens2004.com*

<table>
<thead>
<tr>
<th>Year (1900→0)</th>
<th>Height in.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>75</td>
</tr>
<tr>
<td>12</td>
<td>76</td>
</tr>
<tr>
<td>24</td>
<td>78</td>
</tr>
<tr>
<td>36</td>
<td>80</td>
</tr>
<tr>
<td>56</td>
<td>84</td>
</tr>
<tr>
<td>68</td>
<td>88</td>
</tr>
<tr>
<td>80</td>
<td>93</td>
</tr>
<tr>
<td>88</td>
<td>94</td>
</tr>
<tr>
<td>92</td>
<td>92</td>
</tr>
<tr>
<td>96</td>
<td>94</td>
</tr>
<tr>
<td>100</td>
<td>93</td>
</tr>
<tr>
<td>104</td>
<td>108</td>
</tr>
</tbody>
</table>

32. **Height versus male shoe size:** While it seems reasonable that taller people should have larger feet, there is actually a wide variation in the relationship between height and shoe size. The data in the table show the height (in inches) compared to the shoe size worn for a random sample of 12 male chemistry students. Using the data, (a) draw the scatterplot, (b) determine whether the association is linear or nonlinear, (c) determine whether the association is positive or negative, and (d) find the regression equation and use it to predict the shoe size of a man 80 in. tall and another that is 60 in. tall. Note that the heights of these two men fall outside of the range of our data set (see comment after Example 5 on page 85).

<table>
<thead>
<tr>
<th>Height</th>
<th>Shoe Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>66</td>
<td>8</td>
</tr>
<tr>
<td>69</td>
<td>10</td>
</tr>
<tr>
<td>72</td>
<td>9</td>
</tr>
<tr>
<td>75</td>
<td>14</td>
</tr>
<tr>
<td>74</td>
<td>12</td>
</tr>
<tr>
<td>73</td>
<td>10.5</td>
</tr>
<tr>
<td>71</td>
<td>10</td>
</tr>
<tr>
<td>69.5</td>
<td>11.5</td>
</tr>
<tr>
<td>66.5</td>
<td>8.5</td>
</tr>
<tr>
<td>73</td>
<td>11</td>
</tr>
<tr>
<td>75</td>
<td>14</td>
</tr>
<tr>
<td>65.5</td>
<td>9</td>
</tr>
</tbody>
</table>
33. Plastic money: The total amount of business transacted using credit cards has been changing rapidly over the last 15 to 20 years. The total volume (in billions of dollars) is shown in the table for selected years.

(a) Use a graphing calculator to draw a scatterplot of the data and decide whether the association is linear or nonlinear. (b) Calculate a regression equation with \( x = 0 \) corresponding to 1990 and display the scatterplot and graph on the same screen. (c) According to the equation model, how many billions of dollars were transacted in 2003? How much will be transacted in the year 2011?

Source: Statistical Abstract of the United States, various years

<table>
<thead>
<tr>
<th>( x ) (1990→0)</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>481</td>
</tr>
<tr>
<td>2</td>
<td>539</td>
</tr>
<tr>
<td>4</td>
<td>731</td>
</tr>
<tr>
<td>7</td>
<td>1080</td>
</tr>
<tr>
<td>8</td>
<td>1157</td>
</tr>
<tr>
<td>9</td>
<td>1291</td>
</tr>
<tr>
<td>10</td>
<td>1458</td>
</tr>
<tr>
<td>12</td>
<td>1638</td>
</tr>
</tbody>
</table>

34. Sales of hybrid cars: Since their mass introduction near the turn of the century, the sales of hybrid cars in the United States grew steadily until late 2007, when the price of gasoline began showing signs of weakening and eventually dipped below $3.00/gal. Estimates for the annual sales of hybrid cars are given in the table for the years 2002 through 2009 (2000→0).

(a) Use a graphing calculator to draw a scatterplot of the data and decide if the association is linear or nonlinear. (b) If linear, calculate a regression model for the data and display the scatterplot and data on the same screen. (c) Assuming that sales of hybrid cars recover, how many hybrids does the model project will be sold in the year 2012?

Source: http://www.hybridcar.com

<table>
<thead>
<tr>
<th>Year (2000→0)</th>
<th>Hybrid Sales (in thousands)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>35</td>
</tr>
<tr>
<td>3</td>
<td>48</td>
</tr>
<tr>
<td>4</td>
<td>88</td>
</tr>
<tr>
<td>5</td>
<td>200</td>
</tr>
<tr>
<td>6</td>
<td>250</td>
</tr>
<tr>
<td>7</td>
<td>352</td>
</tr>
<tr>
<td>8</td>
<td>313</td>
</tr>
<tr>
<td>9</td>
<td>292</td>
</tr>
</tbody>
</table>

35. It can be very misleading to rely on the correlation coefficient alone when selecting a regression model. To illustrate, (a) run a linear regression on the data set given (without doing a scatterplot), and note the strength of the correlation (the correlation coefficient). (b) Now run a quadratic regression (press \( \text{CALC} 5: \text{QuadReg} \)) and note the strength of the correlation. (c) What do you notice? What factors other than the correlation coefficient must be taken into account when choosing a form of regression?

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>67</td>
</tr>
<tr>
<td>100</td>
<td>125</td>
</tr>
<tr>
<td>150</td>
<td>145</td>
</tr>
<tr>
<td>200</td>
<td>275</td>
</tr>
<tr>
<td>250</td>
<td>370</td>
</tr>
<tr>
<td>300</td>
<td>550</td>
</tr>
<tr>
<td>350</td>
<td>600</td>
</tr>
</tbody>
</table>

36. In his book Gulliver’s Travels, Jonathan Swift describes how the Lilliputians were able to measure Gulliver for new clothes, even though he was a giant compared to them. According to the text, “Then they measured my right thumb, and desired no more... for by mathematical computation, once around the thumb is twice around the wrist, and so on to the neck and waist.” Is it true that once around the neck is twice around the waist? Find at least 10 willing subjects and take measurements of their necks and waists in millimeters. Arrange the data in ordered pair form (circumference of neck, circumference of waist). Draw the scatterplot for this data. Does the association appear to be linear? Find the equation of the best fit line for this set of data. What is the slope of this line? Is the slope near \( m = 2 \)?
MAINTAINING YOUR SKILLS

37. (1.3) Is the graph shown here, the graph of a function? Discuss why or why not.

38. (Appendix A.2/A.3)
   Determine the area of the figure shown
   \[ A = LW, \quad A = \pi r^2. \]

39. (1.5) Solve for \( r \):
   \[ A = P + Prt \]

40. (Appendix A.3) Solve for \( w \) (if possible):
   \[ -2(6w^2 + 5) - 1 = 7w - 4(3w^2 + 1) \]

MAKING CONNECTIONS

Making Connections: Graphically, Symbolically, Numerically, and Verbally

Eight graphs (a) through (h) are given. Match the characteristics shown in 1 through 16 to one of the eight graphs.

1. \( y = \frac{1}{3}x + 1 \)
2. \( y = -x + 1 \)
3. \( m > 0, \, b < 0 \)
4. \( x = -1 \)
5. \( y = -2 \)
6. \( m < 0, \, b < 0 \)
7. \( m = -2 \)
8. \( m = \frac{2}{3} \)
9. \( f(-3) = 4, \, f(1) = 0 \)
10. \( f(-4) = 3, \, f(4) = 3 \)
11. \( f(x) \geq 0 \) for \( x \in [-3, \, \infty) \)
12. \( x = 3 \)
13. \( f(x) \leq 0 \) for \( x \in [1, \, \infty) \)
14. \( m \) is zero
15. function is increasing, \( y \)-intercept is negative
16. function is decreasing, \( y \)-intercept is negative
SUMMARY AND CONCEPT REVIEW

SECTION 1.1 Rectangular Coordinates; Graphing Circles and Other Relations

KEY CONCEPTS
• A relation is a collection of ordered pairs (x, y) and can be stated as a set or in equation form.
• As a set of ordered pairs, we say the relation is pointwise-defined. The domain of the relation is the set of all first coordinates, and the range is the set of all corresponding second coordinates.
• A relation can be expressed in mapping notation \( x \rightarrow y \), indicating an element from the domain is mapped to (corresponds to or is associated with) an element from the range.
• The graph of a relation in equation form is the set of all ordered pairs (x, y) that satisfy the equation. We plot a sufficient number of points and connect them with a straight line or smooth curve, depending on the pattern formed.
• The x- and y-variables of linear equations and their graphs have implied exponents of 1.
• With a relation entered on the \( \mathbb{C} \) screen, a graphing calculator can provide a table of ordered pairs and the related graph.
• The midpoint of a line segment with endpoints \((x_1, y_1)\) and \((x_2, y_2)\) is \( \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) \).
• The distance between the points \((x_1, y_1)\) and \((x_2, y_2)\) is \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \).
• The equation of a circle centered at \((h, k)\) with radius \( r \) is \((x - h)^2 + (y - k)^2 = r^2\).

EXERCISES
1. Represent the relation in mapping notation, then state the domain and range.
   \[ \{(-7, 3), (-4, -2), (5, 1), (-7, 0), (3, -2), (0, 8)\} \]
2. Graph the relation \( y = \sqrt{25 - x^2} \) by completing the table, then state the domain and range of the relation.

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-5</td>
<td></td>
</tr>
<tr>
<td>-4</td>
<td></td>
</tr>
<tr>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

3. Use a graphing calculator to graph the relation \( 5x + 3y = -15 \). Then use the TABLE feature to determine the value of \( y \) when \( x = 0 \), and the value(s) of \( x \) when \( y = 0 \), and write the results in ordered pair form.

Mr. Northeast and Mr. Southwest live in Coordinate County and are good friends. Mr. Northeast lives at 19 East and 25 North or \((19, 25)\), while Mr. Southwest lives at 14 West and 31 South or \((-14, -31)\). If the streets in Coordinate County are laid out in one mile squares,
4. Use the distance formula to find how far apart they live.
5. If they agree to meet halfway between their homes, what are the coordinates of their meeting place?
6. Sketch the graph of \( x^2 + y^2 = 16 \).
7. Sketch the graph of \( x^2 + y^2 + 6x + 4y + 9 = 0 \).
8. Find an equation of the circle whose diameter has the endpoints \((-3, 0)\) and \((0, 4)\).
SECTION 1.2 Linear Equations and Rates of Change

KEY CONCEPTS

- A linear equation can be written in the form $ax + by = c$, where $a$ and $b$ are not simultaneously equal to 0.
- The slope of the line through $(x_1, y_1)$ and $(x_2, y_2)$ is $m = \frac{y_2 - y_1}{x_2 - x_1}$, where $x_1 \neq x_2$.
- Other designations for slope are $m = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}}$.
- Lines with positive slope ($m > 0$) rise from left to right; lines with negative slope ($m < 0$) fall from left to right.
- The equation of a horizontal line is $y = k$; the slope is $m = 0$.
- The equation of a vertical line is $x = h$; the slope is undefined.
- Lines can be graphed using the intercept method. First determine $(x, 0)$ (substitute 0 for $y$ and solve for $x$), then $(0, y)$ (substitute 0 for $x$ and solve for $y$). Then draw a straight line through these points.
- Parallel lines have equal slopes ($m_1 = m_2$); perpendicular lines have slopes that are negative reciprocals $(m_1 = -\frac{1}{m_2}$ or $m_1 \cdot m_2 = -1$).

EXERCISES

9. Plot the points and determine the slope, then use the ratio $\Delta y \over \Delta x$ to find an additional point on the line:
   a. $(-4, 3)$ and $(5, -2)$ and b. $(3, 4)$ and $(-6, 1)$.
10. Use the slope formula to determine if lines $L_1$ and $L_2$ are parallel, perpendicular, or neither:
    a. $L_1$: $(-2, 0)$ and $(0, 6)$; $L_2$: $(1, 8)$ and $(0, 5)$
    b. $L_1$: $(1, 10)$ and $(-1, 7)$; $L_2$: $(-2, -1)$ and $(1, -3)$
11. Graph each equation by plotting points: (a) $y = 3x - 2$ (b) $y = -\frac{3}{2}x + 1$.
12. Find the intercepts for each line and sketch the graph: (a) $2x + 3y = 6$ (b) $y = \frac{3}{2}x - 2$.
13. Identify each line as either horizontal, vertical, or neither, and graph each line.
    a. $x = 5$ b. $y = -4$ c. $2y + x = 5$
14. Determine if the triangle with the vertices given is a right triangle: $(-5, -4), (7, 2), (0, 16)$.
15. Find the slope and $y$-intercept of the line shown and discuss the slope ratio in this context.

SECTION 1.3 Functions, Function Notation, and the Graph of a Function

KEY CONCEPTS

- A function is a relation, rule, or equation that pairs each element from the domain with exactly one element of the range.
- The vertical line test says that if every vertical line crosses the graph of a relation in at most one point, the relation is a function.
- The domain and range can be stated using set notation, graphed on a number line, or expressed using interval notation.
• On a graph, vertical boundary lines can be used to identify the domain, or the set of “allowable inputs” for a function.
• On a graph, horizontal boundary lines can be used to identify the range, or the set of y-values (outputs) generated by the function.
• When a function is stated as an equation, the implied domain is the set of x-values that yield real number outputs.
• x-values that cause a denominator of zero or that cause the radicand of a square root expression to be negative must be excluded from the domain.
• The phrase “y is a function of x,” is written as $y = f(x)$. This notation enables us to summarize the three most important aspects of a function with a single expression (input, sequence of operations, output).

EXERCISES
16. State the implied domain of each function:
   a. $f(x) = \sqrt{4x + 5}$
   b. $g(x) = \frac{x - 4}{x^2 - x - 6}$

17. Determine $h(-2)$, $h(-\frac{3}{2})$, and $h(3a)$ for $h(x) = 2x^2 - 3x$.

18. Determine if the mapping given represents a function. If not, explain how the definition of a function is violated.

19. For the graph of each function shown, (a) state the domain and range, (b) find the value of $f(2)$, and (c) determine the value(s) of $x$ for which $f(x) = 1$.

SECTION 1.4 Linear Functions, Special Forms, and More on Rates of Change

KEY CONCEPTS
• The equation of a nonvertical line in slope-intercept form is $y = mx + b$ or $f(x) = mx + b$. The slope of the line is $m$ and the y-intercept is $(0, b)$.
• To graph a line given its equation in slope-intercept form, plot the y-intercept, then use the slope ratio $m = \frac{\Delta y}{\Delta x}$ to find a second point, and draw a line through these points.
• If the slope $m$ and a point $(x_1, y_1)$ on the line are known, the equation of the line can be written in point-slope form: $y - y_1 = m(x - x_1)$. 
• A secant line is the straight line drawn through two points on a nonlinear graph.
• The notation \( m = \frac{\Delta y}{\Delta x} \) literally means the quantity measured along the y-axis is changing with respect to changes in the quantity measured along the x-axis.

**EXERCISES**

20. Write each equation in slope-intercept form, then identify the slope and y-intercept.
   a. \( 4x + 3y - 12 = 0 \)
   b. \( 5x - 3y = 15 \)

21. Graph each equation using the slope and y-intercept.
   a. \( f(x) = \frac{7}{3}x + 1 \)
   b. \( h(x) = \frac{5}{2}x - 3 \)

22. Graph the line with the given slope through the given point.
   a. \( m = \frac{3}{5}; (1, 4) \)
   b. \( m = -\frac{1}{2}; (-2, 3) \)

23. What are the equations of the horizontal line and the vertical line passing through \((-2, 5)\)? Which line is the point \((7, 5)\) on?

24. Find the equation of the line passing through \((1, 2)\) and \((-3, 5)\). Write your final answer in slope-intercept form.

25. Find the equation for the line that is parallel to \(4x - 3y = 12\) and passes through the point \((3, 4)\). Write your final answer in slope-intercept form.

26. Determine the slope and y-intercept of the line shown. Then write the equation of the line in slope-intercept form and interpret the slope ratio \( m = \frac{\Delta W}{\Delta R} \) in the context of this exercise.

27. For the graph given, (a) find the equation of the line in point-slope form, (b) use the equation to predict the x- and y-intercepts, (c) write the equation in slope-intercept form, and (d) find y when x = 20, and the value of x for which y = 15.

**SECTION 1.5  Solving Equations and Inequalities Graphically; Formulas and Problem Solving**

**KEY CONCEPTS**

• To use the intersection-of-graphs method for solving equations, assign the left-hand expression as \( Y_1 \) and the right-hand as \( Y_2 \). The solution(s) of the original equation are the x-coordinate(s) of the point(s) of intersection of the graphs of \( Y_1 \) and \( Y_2 \).

• When an equation is written in the form \( h(x) = 0 \), the solutions can be found using the x-Intercept/Zeroes method. Assign \( h(x) \) as \( Y_1 \), and find the x-intercepts of its graph.

• Linear inequalities can be solved by first applying the intersection-of-graphs method to identify the boundary value of the solution interval. Next, the solution is determined by a careful observation of the relative positions of the graphs (is \( Y_1 \) above or below \( Y_2 \)) and the given inequality.

• To solve formulas for a specified variable, focus on the object variable and apply properties of equality to write this variable in terms of all others.

• The basic elements of good problem solving include:
  1. Gathering and organizing information
  2. Making the problem visual
  3. Developing an equation model
  4. Using the model to solve the application

For a complete review, see the problem-solving guide on page 71.
EXERCISES
28. Solve the following equation using the intersection-of-graphs method.
\[3(x - 2) + 10 = 16 - 2(3 - 2x)\]
29. Solve the following equation using the x-intercept/zeroes method.
\[2(x - 1) + \frac{3}{2} = 5\left(\frac{3}{2}x + \frac{1}{2}\right) - \frac{3}{2}\]
30. Solve the following inequality using the intersection-of-graphs method.
\[3(x + 2) - 2.2 < -2 + 4(0.2 - 0.5x)\]

Solve for the specified variable in each formula or literal equation.
31. \[V = \pi r^2 h\] for \( h \)  
32. \[P = 2L + 2W\] for \( L \)
33. \[ax + b = c\] for \( x \)  
34. \[2x - 3y = 6\] for \( y \)

Use the problem-solving guidelines (page 71) to solve the following applications.
35. At a large family reunion, two kegs of lemonade are available. One is 2% sugar (too sour) and the second is 7% sugar (too sweet). How many gallons of the 2% keg must be mixed with 12 gallons of the 7% keg to get a 5% mix?
36. A rectangular window with a width of 3 ft and a height of 4 ft is topped by a semi-circular window. Find the total area of the window.
37. Two cyclists start from the same location and ride in opposite directions, one riding at 15 mph and the other at 18 mph. If their radio phones have a range of 22 mi, how many minutes will they be able to communicate?

SECTION 1.6 Linear Function Models and Real Data

KEY CONCEPTS
- A scatterplot is the graph of all the ordered pairs in a real data set.
- When drawing a scatterplot, be sure to scale the axes to comfortably fit the data.
- If larger inputs tend to produce larger output values, we say there is a positive association.
- If larger inputs tend to produce smaller output values, we say there is a negative association.
- If the data seem to cluster around an imaginary line, we say there is a linear association between the variables.
- If the data clearly cannot be approximated by a straight line, we say the variables exhibit a nonlinear association (or sometimes no association).
- The correlation coefficient \( r \) measures how tightly a set of data points cluster around an imaginary curve. The strength of the correlation is given as a value between \(-1\) and \(1\). Measures close to \(-1\) or \(1\) indicate a very strong correlation. Measures close to \(0\) indicate a very weak correlation.
- We can attempt to model linear data sets using an estimated line of best fit.
- A regression line minimizes the vertical distance between all data points and the graph itself, making it the unique line of best fit.

EXERCISES
38. To determine the value of doing homework, a student in college algebra collects data on the time spent by classmates on their homework in preparation for a quiz. Her data is entered in the table shown. (a) Use a graphing calculator to draw a scatterplot of the data. (b) Does the association appear linear or nonlinear? (c) Is the association positive or negative?

<table>
<thead>
<tr>
<th>( x ) (min study)</th>
<th>( y ) (quiz score)</th>
</tr>
</thead>
<tbody>
<tr>
<td>45</td>
<td>70</td>
</tr>
<tr>
<td>30</td>
<td>63</td>
</tr>
<tr>
<td>10</td>
<td>59</td>
</tr>
<tr>
<td>20</td>
<td>67</td>
</tr>
<tr>
<td>60</td>
<td>73</td>
</tr>
<tr>
<td>70</td>
<td>85</td>
</tr>
<tr>
<td>90</td>
<td>82</td>
</tr>
<tr>
<td>75</td>
<td>90</td>
</tr>
</tbody>
</table>

39. If the association in Exercise 38 is linear, (a) use a graphing calculator to find a linear function that models the relation (study time, grade), then (b) graph the data and the line on the same screen. (c) Does the correlation appear weak or strong?

40. According to the function model from Exercise 39, what grade can I expect if I study for 120 minutes?
1. Solve each equation.
   a. \( \frac{2}{3}x - 5 = 7 - (x + 3) \)
   b. \(-5.7 + 3.1x = 14.5 - 4(x + 1.5)\)
   c. \(P = C + kC; \text{ for } C\)
   d. \(P = 2L + 2W; \text{ for } W\)

2. How much water that is 102°F must be mixed with 25 gal of water at 91°F, so that the resulting temperature of the water will be 97°F.

3. To make the bowling team, Jacques needs a three-game average of 160. If he bowled 141 and 162 for the first two games, what score \(S\) must be obtained in the third game so that his average is at least 160?

4. In the 2009 movie Star Trek (Chris Pine, Zachary Quinto, Zoe Zaldana, Eric Bana), Sulu falls off of the drill platform without a parachute, and Kirk dives off the platform to save him. To slow his fall, Sulu uses a spread-eagle tactic, while Kirk keeps his body straight and arms at his side, to maximize his falling speed. If Sulu is falling at a rate of 180 ft/sec, while Kirk is falling at 250 ft/sec, how long would it take Kirk to reach Sulu, if it took Kirk a full 2 sec to react and dive after Sulu?

5. Two relations here are functions and two are not. Identify the nonfunctions (justify your response).
   a. \(x = y^2 + 2\)
   b. \(y = \sqrt{5 - 2x}\)
   c. \(|y| + 1 = x\)
   d. \(y = x^2 + 2x\)

6. Determine whether the lines are parallel, perpendicular, or neither:
   \(L_1: 2x + 5y = -15\) and \(L_2: y = \frac{3}{5}x + 7\).

7. Graph the line using the slope and \(y\)-intercept: \(x + 4y = 8\).

8. Find the center and radius of the circle defined by \(x^2 + y^2 - 4x + 6y - 3 = 0\), then sketch its graph.

9. After 2 sec, a car is traveling 20 mph. After 5 sec, its speed is 40 mph. Assuming the relationship is linear, find the velocity equation and use it to determine the speed of the car after 9 sec.

10. Find the equation of the line parallel to \(6x + 5y = 3\), containing the point \((2, -2)\). Answer in slope-intercept form.

11. My partner and I are at coordinates \((-20, 15)\) on a map. If our destination is at coordinates \((35, -12)\), (a) what are the coordinates of the rest station located halfway to our destination? (b) How far away is our destination? Assume that each unit is 1 mi.

12. Write the equations for lines \(L_1\) and \(L_2\) shown.

13. State the domain and range for the relations shown on graphs 13(a) and 13(b).

14. For the linear function shown, 
   a. Determine the value of \(W(24)\) from the graph.
   b. What input \(h\) will give an output of \(W(h) = 375\)?
   c. Find a linear function for the graph.
   d. What does the slope indicate in this context?
   e. State the domain and range of \(h\).

15. Given \(f(x) = \frac{2 - x^2}{x^2}\), evaluate and simplify:
   a. \(f\left(\frac{4}{3}\right)\)
   b. \(f(a + 3)\)
16. In 2007, there were 3.3 million Apple iPhones sold worldwide. By 2009, this figure had jumped to approximately 30.3 million [Source: http://brainstormtech.blogs.fortune.cnn.com/2009/03/12/]. Assume that for a time, this growth could be modeled by a linear function. (a) Determine the rate of change \( \frac{\Delta \text{sales}}{\Delta \text{time}} \), and (b) interpret it in this context. Then use the rate of change to (c) approximate the number of sales in 2008, and what the projected sales would be for 2010 and 2011.

17. Solve the following equations using the \( x \)-intercept/zeros method.
   \[ 2x + \left(4 - \frac{1}{3}x\right) = -(20 + x) \]
   \[ 2(0.7x - 1.3) + 2.6 = 2x - 3(0.2x - 2) \]

18. Solve the following inequalities using the intersection-of-graphs method.
   \[ 3x - (5 - x) \geq 2(5 - x) + 3 \]
   \[ 2(0.75x - 1) < 0.7 + 0.5(3x - 1) \]

19. To study how annual rainfall affects the ability to attain certain levels of livestock production, a local university collects data on the average annual rainfall for a particular area and compares this to the average number of free-ranging cattle per acre for ranchers in that area. The data collected are shown in the table. (a) Use a graphing calculator to draw a scatterplot of the data. (b) Does the association appear linear or nonlinear? (c) Is the association positive or negative?

<table>
<thead>
<tr>
<th>Rainfall (in.)</th>
<th>Cattle per Acre</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>19</td>
<td>7</td>
</tr>
<tr>
<td>23</td>
<td>9</td>
</tr>
<tr>
<td>28</td>
<td>11</td>
</tr>
<tr>
<td>32</td>
<td>22</td>
</tr>
<tr>
<td>37</td>
<td>23</td>
</tr>
<tr>
<td>40</td>
<td>26</td>
</tr>
</tbody>
</table>

20. If the association in Exercise 19 is linear, (a) use a graphing calculator to find a linear function that models the relation (rainfall, cattle per acre), (b) use the function to find the number of cattle per acre that might be possible for an area receiving 50 in. of rainfall per year, and (c) state whether the correlation is weak or strong.

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**STRENGTHENING CORE SKILLS**

**The Various Forms of a Linear Equation**

Learning mathematics is very much like the construction of a skyscraper. The final height of the skyscraper ultimately depends on the strength of the foundation and quality of the frame supporting each new floor as it is built. Our previous work with linear functions and their graphs, while having a number of useful applications, is actually the foundation on which much of our future work is built. For this reason, it's important you gain a certain fluency with linear functions and relationships—even to a point where things come to you effortlessly and automatically. As noted mathematician Henri Lebesgue once said, "An idea reaches its maximum level of usefulness only when you understand it so well that it seems like you have always known it. You then become incapable of seeing the idea as anything but a trivial and immediate result." These formulas and concepts, while simple, have an endless number of significant and substantial applications.

**Forms and Formulas**

- **slope formula**
  \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]
given any two points on the line

- **point-slope form**
  \[ y - y_1 = m(x - x_1) \]
given slope \( m \) and any point \((x_1, y_1)\)

- **slope-intercept form**
  \[ y = mx + b \]
given slope \( m \) and \( y \)-intercept \((0, b)\)

- **standard form**
  \[ Ax + By = C \]
\( A, B, \) and \( C \) are integers (used in linear systems)
**Characteristics of Lines**

<table>
<thead>
<tr>
<th>y-intercept</th>
<th>x-intercept</th>
<th>increasing</th>
<th>decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, (y))</td>
<td>((x, 0))</td>
<td>(m &gt; 0)</td>
<td>(m &lt; 0)</td>
</tr>
<tr>
<td>let (x = 0), solve for (y)</td>
<td>let (y = 0), solve for (x)</td>
<td>line slants upward from left to right</td>
<td>line slants downward form left to right</td>
</tr>
</tbody>
</table>

**Relationships between Lines**

<table>
<thead>
<tr>
<th>intersecting</th>
<th>parallel</th>
<th>perpendicular</th>
<th>dependent</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_1 \neq m_2)</td>
<td>(m_1 = m_2), (b_1 \neq b_2)</td>
<td>(m_1m_2 = -1)</td>
<td>(m_1 = m_2), (b_1 = b_2)</td>
</tr>
<tr>
<td>lines intersect at one point</td>
<td>lines do not intersect</td>
<td>lines intersect at right angles</td>
<td>lines intersect at all points</td>
</tr>
</tbody>
</table>

**Special Lines**

<table>
<thead>
<tr>
<th>horizontal</th>
<th>vertical</th>
<th>identity</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y = k)</td>
<td>(x = h)</td>
<td>(y = x)</td>
</tr>
<tr>
<td>horizontal line through (k)</td>
<td>vertical line through (h)</td>
<td>the input value identifies the output</td>
</tr>
</tbody>
</table>

Use the formulas and concepts reviewed here to complete the following exercises.

For the two points given: (a) compute the slope of the line through the points and state whether the line is increasing or decreasing, (b) find the equation of the line in point-slope form, then write the equation in slope-intercept form, and (c) find the \(x\)- and \(y\)-intercepts and graph the line.

**Exercise 1:** \(P_1(0, 5)\) \(P_2(6, 7)\)

**Exercise 2:** \(P_1(3, 2)\) \(P_2(0, 9)\)

**Exercise 3:** \(P_1(3, 2)\) \(P_2(9, 5)\)

**Exercise 4:** \(P_1(-5, -4)\) \(P_2(3, 2)\)

**Exercise 5:** \(P_1(-2, 5)\) \(P_2(6, -1)\)

**Exercise 6:** \(P_1(2, -7)\) \(P_2(-8, -2)\)

---

**Evaluating Expressions and Looking for Patterns**

These "explorations" are designed to explore the full potential of a graphing calculator, as well as to use this potential to investigate patterns and discover connections that might otherwise be overlooked. In this exploration and discovery, we point out the various ways an expression can be evaluated on a graphing calculator. Some ways seem easier, faster, and/or better than others, but each has advantages and disadvantages depending on the task at hand, and it will help to be aware of them all for future use.

One way to evaluate an expression is to use the TABLE feature of a graphing calculator, with the expression entered as \(Y_1\) on the \(Y=\) screen. If you want the calculator to generate inputs, use the \(\text{TBLSET}\) (TBLSET) screen to indicate a starting value (\(\text{TblStart} = \)) and an increment value (\(\text{ATbl} = \)), and set the calculator in \(\text{Indpnt: AUTO}\) mode (to input specific values, the calculator should be in \(\text{Indpnt: AUTO ASK}\) mode). After pressing \(\text{TABLE}\), the calculator shows the corresponding input and output values.

Expressions can also be evaluated on the home screen for a single value or a series of values. Enter the expression \(-\frac{3}{4}x + 5\) on the \(Y=\) screen (see Figure 1.1.05) and use \(\text{QUIT}\) to get back to the home screen. To evaluate this expression, access \(Y_1\) using \(\text{VAR\(S\)}\) \(\text{Y-VARS}\), and use the first option \(1: \text{Function}\) \(\text{\(Y=\)}\). This brings us to a submenu where any of the equations \(Y_1\) through \(Y_6\) (actually \(Y_{10}\)) can be accessed. Since the default setting is the one we need \((1: Y_1)\), simply press \(\text{\(Y_1\)}\) and \(Y_1\) appears on the home screen. To evaluate a single input, simply enclose it in

Figure 1.1.05

```
<table>
<thead>
<tr>
<th>Y_1</th>
<th>Y_2</th>
<th>Y_3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3/4)x+5</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

\(Y_1\)
parentheses. To evaluate more than one input, enter the numbers as a set of values with
the set enclosed in parentheses. In Figure 1.106, \( Y_1 \) has been evaluated for \( x = -4 \),
then simultaneously for \( x = -4, -2, 0, \) and \( 2 \).

A third way to evaluate expressions is using a list, with the desired inputs entered
in List 1 (L1), then List 2 (L2) defined in terms of L1. For example, \( L2 = -\frac{3}{4}L1 + 5 \)
will return the same values for inputs of \( -4, -2, 0, \) and \( 2 \) seen previously on the home
screen (remember to clear the lists first). Lists are accessed by pressing \( \text{2nd } \text{Edit} \).

Enter the numbers \( -4, -2, 0, \) and \( 2 \) in L1, then use the right arrow \( \uparrow \) to move to L2. It
is important to note that you next press the up arrow key \( \uparrow \) so that the cursor overlies
L2. The bottom of the screen now reads “L2 = ” and the calculator is waiting for us to
define L2. After entering \( L2 = -\frac{3}{4}L1 + 5 \) (see Figure 1.107) and pressing \( \text{Enter} \) we
obtain the same outputs as before (see Figure 1.108). The advantage of using the “list”
method is that we can further explore or experiment with the output values in a search
for patterns.

**Exercise 1:** Evaluate the expression \( 0.2L1 + 3 \) on the list screen, using consecutive
integer inputs from \(-6\) to \(6\) inclusive. What do you notice about the outputs?

**Exercise 2:** Evaluate the expression \( \sqrt{2}L1 - \sqrt{9} \) on the list screen, using
consecutive integer inputs from \( -6 \) to \( 6 \) inclusive. We suspect there is a pattern to the
output values, but this time the pattern is very difficult to see. On the home screen,
compute the difference between a few successive outputs from L2 (for example,
\( L2(1) - L2(2) \)). What do you notice?
Understanding and internalizing concepts related to linear functions is one of the main objectives of Chapter 1 and its Strengthening Core Skills feature. The ability to quickly and correctly write the equation of a line given sufficient information has a number of substantial applications in the calculus sequence. In calculus, we will extend our understanding of secant lines to help understand lines drawn tangent to a curve and further to lines and planes drawn tangent to three-dimensional surfaces. These relationships will lead to significant and meaningful applications in topography, meteorology, engineering, and many other areas.

**Tangent Lines**

In Section 1.4 we learned how to write the equation of a line given its slope and any point on the line. In that section, Example 6 uses the slope-intercept form \( y = mx + b \) as a formula, while Example 10 applies the point-slope form \( y - y_1 = m(x - x_1) \). In calculus, we regularly use both forms to write the equations of secant lines and tangent lines.

Consider the graph of \( f(x) = x^2 - 2x - 3 \), a parabola that opens upward, with \( y \)-intercept \((0, -3)\) and vertex at \((1, -4)\). Using the tools of calculus, we can show that the function \( g(x) = 2x - 2 \) gives the slope of any line drawn tangent to this graph at a given \( x \). For example, to find the slope of the line tangent to this curve when \( x = 3 \), we evaluate \( g(x) \) at \( x = 3 \), and find the slope will be \( g(3) = 4 \). Note how this information is used in Example 1.

**Example 1**

**Finding the Equation of a Tangent Line**

Find an equation of the line drawn tangent to the graph of \( f(x) = x^2 - 2x - 3 \) at \( x = 0 \), and write the result in slope-intercept form.

**Solution**

To begin, we first determine the slope of the tangent line. For \( x = 0 \), we have

\[
\begin{align*}
g(x) &= 2x - 2 & \text{given equation and slope of tangent line} \\
g(0) &= 2(0) - 2 & \text{substitute } 0 \text{ for } x \\
&= -2 & \text{result}
\end{align*}
\]

The slope of the line drawn tangent to this curve at \( x = 0 \), is \( m = -2 \). However, recall that to find the equation of this line we also need a point on the line. Using \( x = 0 \) once again, we note \((0, -3)\) is the \( y \)-intercept of the graph of \( f(x) \) and is also a point on the tangent line. Using the point \((0, -3)\) and the slope \( m = -2 \), we have

\[
\begin{align*}
y - y_1 &= m(x - x_1) & \text{point-slope form} \\
y - (-3) &= -2(x - 0) & \text{substitute } -2 \text{ for } m, (0, -3) \text{ for } (x, y) \\
y + 3 &= -2x & \text{simplify} \\
y &= -2x - 3 & \text{slope-intercept form}
\end{align*}
\]

In the figure shown, note the graph of \( y = -2x - 3 \) is tangent to the graph of \( y = x^2 - 2x - 3 \) at the point \((0, -3)\).

**Now try Exercises 1 through 8**

Similar to our work here with tangent lines, many applications of advanced mathematics require that we find the equation of a line drawn perpendicular to this tangent line. In this case, the line is called a normal to the curve at point \((x, y)\). See Exercises 9 through 12.
Connections to Calculus Exercises

For Exercises 1 through 4, the function \( g(x) = 2x + 4 \) gives the slope of any line drawn tangent to the graph of \( f(x) = x^2 + 4x - 5 \) at a given \( x \). Find an equation of the line drawn tangent to the graph of \( f(x) \) at the following \( x \)-values, and write the result in slope-intercept form.

1. \( x = -4 \)
2. \( x = 0 \)
3. \( x = -2 \)
4. \( x = 1 \)

For Exercises 5 through 8, the function \( v(x) = 6x^2 - 6x - 36 \) gives the slope of any line drawn tangent to the graph of \( s(x) = 2x^3 - 3x^2 - 36x \) at given \( x \). Find an equation of the line drawn tangent to the graph of \( s(x) \) at the following \( x \)-values, and write the result in slope-intercept form.

5. \( x = -2 \)
6. \( x = 3 \)
7. \( x = 1 \)
8. \( x = 4 \)

For Exercises 9 through 12, find the equation of the normal to each tangent shown, or the equation of the tangent line for each normal given.

12. \( y = -x + 4 \)

13. A theorem from elementary geometry states, “A line drawn tangent to a circle is perpendicular to the radius at the point of tangency.” Find the equation of the tangent line shown in the figure.

14. Find the equation of the line containing the perpendicular bisector of the line segment with endpoints \((-2, -3)\) and \((4, 1)\) shown.