9-1 Mathematical Patterns

A **sequence** is an ordered list of numbers. Each number in a sequence is a **term of a sequence**. You can represent a term of a sequence by using a variable with a subscript number to indicate its position in the sequence. For example, a_5 is the fifth term in the sequence $a_1, a_2, a_3, a_4, \ldots$.

The subscripts of sequence terms are often positive integers starting with 1. If so, you can generalize a term as a_n, the nth term in the sequence.

1st term	2nd term	3rd term	...	$n-1$ term	nth term	$n+1$ term	...
↓	↓	↓		↓	↓	↓	
a_1	a_2	a_3	...	a_{n-1}	a_n	a_{n+1}	...

An **explicit formula** describes the nth term of a sequence using the number n.

For example, in the sequence 2, 4, 6, 8, 10, ..., the nth term is twice the value of n. You write this as $a_n = 2n$. The table shows how to find a_n by substituting the value of n into the explicit formula.

n	nth term
1	$a_1 = 2(1) = 2$
2	$a_2 = 2(2) = 4$
3	$a_3 = 2(3) = 6$
4	$a_4 = 2(4) = 8$

Problem 1 Generating a Sequence Using an Explicit Formula

Plan

How does the explicit formula help you find the value of a term? Replace n in the formula with the number of the term. Simplify to find the value of the term.

A sequence has an explicit formula $a_n = 3n - 2$. What are the first 10 terms of this sequence?

$a_n = 3n - 2$ Write the formula.

$a_1 = 3(1) - 2 = 1$ Substitute 1 for n and simplify.

$a_2 = 3(2) - 2 = 4$ Substitute 2 for n and simplify.

You can use a table to organize your work for the remaining terms.

n	a_n	
3	$a_3 = 3(3) - 2 = 7$	Substitute 3 for n and simplify.
4	$a_4 = 3(4) - 2 = 10$	
5	$a_5 = 3(5) - 2 = 13$	And so on.
6	$a_6 = 3(6) - 2 = 16$	
7	$a_7 = 3(7) - 2 = 19$	
8	$a_8 = 3(8) - 2 = 22$	
9	$a_9 = 3(9) - 2 = 25$	
10	$a_{10} = 3(10) - 2 = 28$	

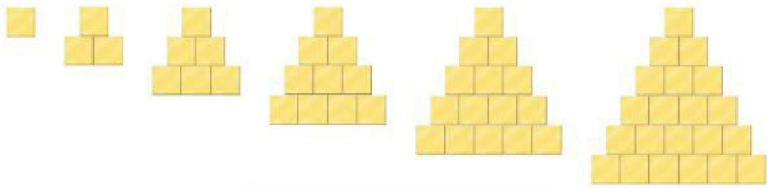
The first ten terms are 1, 4, 7, 10, 13, 16, 19, 22, 25, 28.

Sometimes you can see the pattern in a sequence by comparing each term to the one that came before it. For example, in the sequence 133, 130, 127, 124, . . . , each term after the first term is equal to three less than the previous term.

A recursive definition for this sequence contains two parts.
(a) an initial condition (the value of the first term): $a_1 = 133$
(b) a **recursive formula** (relates each term after the first term to the one before it):
$a_n = a_{n-1} - 3$, for $n > 1$

Problem 2 Writing a Recursive Definition for a Sequence

The number of blocks in a two-dimensional pyramid is a sequence that follows a recursive formula. What is a recursive definition for the sequence?

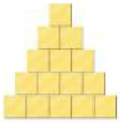


Think	Write
Count the number of blocks in each pyramid.	1, 3, 6, 10, 15, 21
Subtract consecutive terms to find out what happens from one term to the next.	$a_2 - a_1 = 3 - 1 = 2$ $a_3 - a_2 = 6 - 3 = 3$ $a_4 - a_3 = 10 - 6 = 4$ $a_5 - a_4 = 15 - 10 = 5$ $a_6 - a_5 = 21 - 15 = 6$
Use n to express the relationship between successive terms.	$a_n - a_{n-1} = n$
To write a recursive definition, state the initial condition and the recursive formula.	$a_1 = 1$ and $a_n = a_{n-1} + n$.

Got It? **2.** What is a recursive definition for each sequence?
(*Hint:* Look for simple addition or multiplication patterns to relate consecutive terms.)
a. 1, 2, 6, 24, 120, 720, . . .
b. 1, 5, 14, 30, 55, . . .

Problem 3 Writing an Explicit Formula for a Sequence

What is the 100th term of the pyramid sequence in Problem 2?



Plan

Why do you use the explicit formula to find a_{100}?
Because starting with a_1, it would take 99 iterations of the formula to get a_{100} using the recursive formula.

To find an explicit formula, expand the first few terms of the pyramid sequence.

a_1	a_2	a_3	a_4	a_5	...	a_n
1	1 + 2	1 + 2 + 3	1 + 2 + 3 + 4	1 + 2 + 3 + 4 + 5	...	1 + 2 + ... + n
1	3	6	10	15	...	■

Therefore,

$$a_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n,$$

which you can write as

$$a_n = n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.$$

Adding the two previous equations gives the following result:

$$\begin{aligned} a_n &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\ + a_n &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\ \hline 2a_n &= (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1) \\ 2a_n &= n(n+1) \\ a_n &= \tfrac{1}{2}n(n+1) \end{aligned}$$

The explicit formula for this sequence is $a_n = \frac{1}{2}n(n+1)$.

Substitute 100 into the explicit formula to find the 100th term.

$$\begin{aligned} a_{100} &= \tfrac{1}{2}(100)(100+1) \\ &= \tfrac{1}{2}(100)(101) \\ &= 5050 \end{aligned}$$

The 100th term is 5050.